Ph.D. Comprehensive Examination on Algebra

Fall 2003

You have three hours to complete this exam. Please write your solutions in a clear and concise fashion.

1. Suppose R is a commutative ring with identity where $1_R \neq 0$. Prove that the following are equivalent.

   (i) R is a field;

   (ii) R has no proper ideals;

   (iii) 0 is a maximal ideal in R;

   (iv) every nonzero homomorphism of rings $R \to S$ is a monomorphism.

2. State the three Sylow theorems. Prove that there are no simple groups of order 36.

3. Compute the Galois group of $x^4 + 1$ over $\mathbb{Q}$.

4. Let A be a symmetric real $n \times n$ matrix. Show that all the eigenvalues of A must be real and that the eigenvectors corresponding to the different eigenvectors are orthogonal.

5. Determine the Jordan canonical form of the following matrix:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

6. Prove that every group of order $p^2 q$ where p and q are primes is solvable.

7. Let K be a field. Prove that the polynomial ring $K[x]$ is a principal ideal domain.

8. Let R be a ring and $f : M \to N$ and $g : N \to M$ be R-module homomorphisms such that $g \circ f = \mathrm{id}_M$. Show that $N \cong \mathrm{Im} f \oplus \mathrm{Ker}\, g$.