PH.D. PROBABILITY PRELIMINARY EXAMINATION

Fall 2002

(1) (a) Quote, without proof, the Kolmogorov zero-one law.

(b) Let $\{a_n\}$ be any sequence of real numbers and $\{X_n\}$ be a sequence of independent random variables taking the values ± 1 with equal probability. Show that the convergence set $C := \{\omega : \sum a_n X_n \textit{ converges}\}$ is a tail event.

(2) (a) Let $\{X_n,\ n \geq 1\}$ be a sequence of pairwise independent random variables with $P\{X_n = n^\delta\} = \frac{1}{2} = P\{X_n = -n^\delta\}$, where $0 < \delta < \frac{1}{2}$. Does the law of large numbers (weak or strong) hold for this sequence?

(b) Let $\{X_n,\ n \geq 1\}$ be a sequence of independent random variables with $P\{X_n = \sqrt{n}\} = \frac{1}{2} = P\{X_n = -\sqrt{n}\}$. Does the weak law of large numbers hold for this sequence?

(3) Let $f : (0, \infty) \to \mathbb{R}$ be a bounded continuous function. Prove that the limit relation

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} f\left(x + \frac{k}{n}\right) \exp(-nh) \frac{(nh)^k}{k!} = f(x+h),$$

holds for all $h > 0,\ x > 0$. [Hint: Independent Poisson *r.v.*s.]

(4) Let $F_n,\ F,\ n \geq 1$, be distribution functions such that F_n converges weakly (*i.e.*, in distribution) to F as $n \to \infty$.

(a) If the function $g : \mathbb{R} \to \mathbb{R}$ is uniformly integrable with respect to F_n, show that $\int g\, dF_n \to \int g\, dF$.

(b) If $\int |g|\, dF_n \to \int |g|\, dF < \infty$, show that g is uniformly integrable.

(5) Let $\{X_n\}$, $n \geq 1$, be a sequence of independent random variables and $\{a_n\}$, $n \geq 1$, be a sequence of real numbers such that $P(X_n = a_n) = \frac{1}{2} = P(X_n = -a_n)$, $n \geq 1$. Find conditions on $\{a_n\}$ so that the sequence $\{X_n\}$ will satisfy the central limit property.

(6) (a) Let $\{X_n\}$, $n \geq 1$, be a sequence of nonnegative and uniformly bounded random variables adapted to an increasing sequence $\{\mathcal{F}_n,\ n \geq 1\}$ of sub σ-algebras. Show that the series

$$\sum_{n=1}^{\infty} X_n, \quad \text{and} \quad \sum_{n=1}^{\infty} E\{X_n \mid \mathcal{F}_{n-1}\},$$

where $\mathcal{F}_0$ is the trivial σ-algebra, either both converge *a.s.* or both diverge *a.s.*.

(b) Let $\{\mathcal{F}_n,\ n \geq 1\}$ be an increasing sequence of σ-algebras and $A_n \in \mathcal{F}_n,\ n \geq 1$ and write $p_1 = P(A_1)$ and $p_n = P\{A_n \mid \mathcal{F}_{n-1}\}$. Show that

$$P(\limsup_n A_n) = 1 \quad \text{if and only if} \quad P\left\{\omega : \sum_{n=1}^{\infty} p_n(\omega) = \infty\right\} = 1.$$

(7) Let $\{X_n,\ n \geq 1\}$ be a sequence of *i.i.d* random variables with zero mean and unit variance, $S_n := X_1 + \cdots + X_n,\ n \geq 1$, and let $\{\nu_n,\ n \geq 1\}$ be a sequence of positive integer valued random variables converging to ∞ in probability as $n \to \infty$. Assume that the two sequences $\{X_n,\ n \geq 1\}$ and $\{\nu_n,\ n \geq 1\}$ are independent of each other. Show that

$$\lim_{n\to\infty} P\{S_{\nu_n} < x\} = N(0,1),$$

where $N(0,1)$ is the standard normal distribution. [Hint: You may need Kronecker lemma.]