P.H.D. Preliminary Examination Numerical Analysis
August 13, 2001

Do 10 of the 12 problems

1) Suppose that r is a double zero of the function f. Show that if f'' is continuous, then Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ converges linearly to r with $e_{n+1} \approx \frac{1}{2}e_n$.

2)(i) Define the Chebyshev polynomial of the first kind, $T_n(x)$, of degree n.
(ii) Find a formula for the n zeros, $z_1, z_2, \ldots, z_n$, of $T_n(x)$ in $(-1,1)$.
(iii) Find a formula for the $n+1$ points $x_0, x_1, \ldots, x_n$ where $T_n(x)$ alternates between 1 and -1, which are its extreme values on $[-1,1]$.
(iv) Let $P_n(x)$ be any monic polynomial of degree n. Show that

$$\max_{-1\le x\le 1} |P_n(x)| \ge \max_{-1\le x\le 1} |2^{1-n}T_n(x)| = 2^{1-n}$$

3) When solving the quadratic equation $ax^2 + bx + c = 0$ by use of the formulas

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

there will be a loss of significant digits, due to subtractive cancellation, in one of these formulas when $4ac$ is small relative to b^2. Find a way around this problem.

4) Given the table

x	$f(x)$
0.05	2.857651118
0.025	2.787095461
−0.025	2.651167211
−0.05	2.585709659

compute your best possible estimate of $f'(0)$. Carefully explain why your estimate is the best possible estimate of $f'(0)$.

5) (i) Define the condition number, $K(A)$, of a $N \times N$ matrix A relative to a subordinate matrix norm $\| \ \|$.
(ii) If $\tilde{x}$ is an approximate solution of the equation $Ax = b$, then carefully derive the relative error estimate

$$\frac{1}{K(A)}\frac{\|r\|}{\|b\|} \le \frac{\|x - \tilde{x}\|}{\|x\|} \le K(A)\frac{\|r\|}{\|b\|}$$

with residual vector $r = b - A\tilde{x}$.
(iii) Suppose that $K(A) = 10^{10}$ and that you intend to solve the equation $Ax = b$ on a computer that does floating point arithmetic to approximately 16 significant decimal digits. What does the above relative error estimate say the about the accuracy of any solution computed on this computer?

6) Show that for any $N \times N$ matrix T, and any initial guess x_0, the iteration

$$x_{k+1} = Tx_k + c$$

converges to a solution of the equation

$$x = Tx + c$$

if and only if the spectral radius of T is less than 1.

7) (i) Determine values of a, b, c so that the function $S(x)$ defined below is a C^2 cubic spline:

$$S(x) = \begin{cases} 1 + x^3 & \text{if } 0 \le x \le 1 \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 & \text{if } 1 < x \le 2 \end{cases}$$

(ii) Next, using the values of a, b, c found in (i), determine d so that $\int_0^1 S''(x)^2 dx$ is a minimum.
(iii) Finally, using the values of a, b, c found in (i), determine d so that $S(x)$ is a natural cubic spline.

8) (i) Find constants A and c so that the approximation

$$\int_0^1 x^4 f(x) dx \approx Af(c)$$

is exact for $f(x)$ a polynomial of degree ≤ 1.
(ii) Using the values of A and c found in (i) find a constant K so that for all $f \in C^2([0,1])$ the error E in the above approximation satifies

$$|E| \le K \max_{0 \le x \le 1} |f''(x)|$$

9) Use the Cholesky decomposition to prove that these two properties of a n by n symmetric matrix A are equivalent:
(i) A is positive definite
(ii) There exists a linearly independent set of vectors $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n}$ in R^n such that $A_{i,j} = (\vec{x_i})^t(\vec{x_j})$

10) Show that if the n by n matrix A is symmetric and positive definite, then the problem of solving $Ax = b$ is equivalent to the problem of minimizing the quadratic form

$$q(x) = \langle x, Ax \rangle - 2\langle x, b \rangle$$

11) (i) Use a generalized divided difference table to determine the polynomial $P(x)$ of degree ≤ 4 interpolating the data in the following table:

x	$f(x)$	$f'(x)$
0	2	−9
1	−4	4
2	44	

(ii) Assume that $\max_{0 \le x \le 2} |f^{(v)}(x)| = 1$. Find an upper bound on $\max_{0 \le x \le 2} |f(x) - P(x)|$.

12) Given weight function $w(x) > 0$ on $[a, b]$ and corresponding inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx$$

show that the sequence of polynomials defined inductively as follows is orthogonal:

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x) \quad (n \geq 2)$$

with $p_0(x) = 1$, $p_1(x) = x - a_1$, and

$$a_n = \frac{\langle xp_{n-1}, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle}$$

$$b_n = \frac{\langle xp_{n-1}, p_{n-2} \rangle}{\langle p_{n-2}, p_{n-2} \rangle}$$