Qualifying Exam Algebra

Answer eight of the following ten questions.

1. Let G be the subgroup $\{e^{it} \mid t \in \mathbb{R}\}$ of the multiplicative group $\mathbb{C} - \{0\}$.

   (a) Show that the subset H of G formed by all the elements of finite order is an infinite muliplicative group with infinite exponent.

   (b) Show that H is isomorphic to the (additive) group $\mathbb{Q}/\mathbb{Z}$.

   (c) Show that any finite subgroup of H (or equivalently of $\mathbb{Q}/\mathbb{Z}$) is cyclic.

2. Let p be an odd prime number and consider the set $G = \{(x, y, z) \in (\mathbb{Z}/p\mathbb{Z})^3\}$. Using the usual addition and multiplication in $\mathbb{Z}/p\mathbb{Z}$, define the composition law $*$ on G :

$$(x, y, z) * (x', y', z') = (x + x', y + y', xy' + z + z') \quad \text{for all } (x, y, z), (x', y', z') \in G .$$

   (a) Show that G is isomorphic to a Sylow p-subgroup of the (multiplicative) group $\mathrm{GL}_3(p)$ of invertible 3×3 matrices with coefficients in the field $\mathbb{Z}/p\mathbb{Z}$ via the map:

$$(x, y, z) \longmapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

   (b) Find the center and the commutator subgroups of G .

3. Let R be a commutative ring (with identity 1) and let I be an ideal of R that is contained in all the maximal ideals of R . Show that $1 - x$ is a unit, for all $x \in I$.

4. Let R be the ring $\mathbb{Z}[i]$ of Gaussian integers.

   (a) Factorize $9 + 19i$ as product of irreducibles of R .

   (b) Determine the field of fractions F of R .

   (c) Show that $f = 3X^3 - 6X^2 + 12X - 6$ is irreducible in F but not in R .

5. Let R be a (non necessarily commutative) ring (with identity 1) and let $e \in R$ be a central idempotent, that is $e^2 = e$ and e is an element of the center of R . Let M be a left R-module and set $eM = \{e \cdot x \mid x \in M\}$ and $(1 - e)M = \{(1 - e) \cdot x \mid x \in M\}$.

   (a) Show that eM and $(1 - e)M$ are R-submodules of M .

   (b) Show that there is a split short exact sequence

$$0 \longrightarrow eM \xrightarrow{f} M \xrightarrow{g} (1 - e)M \longrightarrow 0$$

   of left R-modules, where f is the inclusion and g is the left multiplication by $(1 - e)$.

6. Let R be an integral domain in which there exists no sequence $(a_n)_{n\in\mathbb{N}}$ such that a_{n+1} is a proper factor of a_n (that is a_{n+1} divides a_n and $a_{n+1} \neq a_n$). Prove that R is a unique factorization domain (UFD) if and only if any irreducible element of R is also prime.

7. Let F be a field, E be a field extension of F, and K be a field extension of E.

   (a) Show that if the extension K/F is separable then the extensions E/F and K/E are also separable.

   (b) Is the converse true? (no proof required)

8. Let k be a field and let V be a finite dimensional k-vector space. Denote by V^* the dual of V, that is the set of all homomorphisms of k-vector spaces from V to k.

   (a) Show that the following map, defined on a set of generators of $V^* \otimes V$, extends to a surjective homomorphism of k-vector spaces:

   $$t : V^* \otimes V \longrightarrow k, \quad t(\varphi \otimes v) = \varphi(v)$$

   (b) Show that t admits a right inverse if and only if the characteristic of k is either zero, or does not divide the dimension of V.

9. Alex, Bart and Carl do their laundry at the same location. Alex washes his clothes once every 11 days, Bart one Friday each 2 weeks, and Carl once every 5 days. Last time that Alex did his laundry was on December 29, 2004; whereas for Bart it was on Friday December 31, 2004; and for Carl it was on January 1st, 2005.

   After how many days will/did all 3 of them wash their clothes on the same day, for the first time after January 1st, 2005? (that is 01/01/05 is day 1)

10. Let

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 0 & -1 & -4 \\ -4 & 0 & 0 & 3 \end{pmatrix} \in M_4(\mathbb{R})$$

   (a) Find the Jordan form J of A.

   (b) Find an invertible matrix P such that $J = P^{-1}AP$. (Note that you do not need to compute P^{-1})

   (c) Find the minimal polynomial of A.