Algebra Prelim: Fall 1998

Instructions: Attempt all problems. The number of completed problems is important; one complete problem is worth more than two half-done problems.

1. Let A be a hermitian $n \times n$ matrix over $\mathbb{C}$. Give a self-contained proof that there exists a matrix U such that UAU^{-1} is diagonal.

2. (a) State what it means for a matrix to be in Jordan form.
   (b) Give the Jordan form of the matrix
   $$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}.$$
   (c) What is the characteristic polynomial of the matrix in (b)? The minimal polynomial?

3. Let $f(x) = x^4 - 5$.
   (a) Describe the Galois group of f over $\mathbb{Q}$.
   (b) Let K be a splitting field for f over $\mathbb{Q}$. What is $[K : \mathbb{Q}]$?
   (c) How many intermediate fields are there between K and $\mathbb{Q}$ (inclusive)?

4. (a) Let F be a field and let $f(x) \in F[x]$ be a monic polynomial of positive degree. Show that there exists a field extension $K \supset F$ such that $f(x)$ factors into linear factors in $K[x]$.
   (b) Show that $x^{25} - x$ has no multiple roots in a field of characteristic 5.
   (c) Show that there exists a field with exactly 25 elements.

5. (a) State the 3 Sylow Theorems.
   (b) What is the order of $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$?
   (c) Exhibit a Sylow 5-subgroups of $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$.

6. (a) State the fundamental theorem of finitely generated modules over a PID.
   (b) Apply this theorem to give a proof of the classification of finite abelian groups up to isomorphism.

7. In this problem and the next, rings are commutative with 1.

   (a) Give an example of a ring A, a maximal ideal $\mathfrak{m}$, and a nonzero A-module M such that $M_{\mathfrak{m}}$ is the zero module.

   (b) Let A be a ring and M a nonzero A-module. Show that for any nonzero element $x \in M$, there exists a maximal ideal $\mathfrak{m}$ such that $x/1 \in M_{\mathfrak{m}}$ is nonzero. Conclude that for any nonzero module M, there is a maximal ideal $\mathfrak{m}$ such that $M_{\mathfrak{m}} \neq 0$.

8. Let A be a local domain with maximal ideal $\mathfrak{m}$

   (a) State Nakayama's lemma for A.

   (b) Let M be a finitely generated A-module. Show that the minimal number of generators for M is $\dim_{A/\mathfrak{m}} M/\mathfrak{m}M$.

9. Let G be a finite group, p the smallest prime dividing the order of G, and H a subgroup of index p. Show that H is normal. Hint: Consider the action of G on G/H.