Preliminary Exam in Algebra

Sept. 1997

Do as many problems as you can; each problem is worth 10 points. The number of problems done **completely** will also be taken into account: one correct problem is better than two half-done problems.

1. Let $f(x) = x^3 - 4x + 2 \in \mathbb{Q}[x]$.
   (a) Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$.
   (b) Determine the Galois group G of f over $\mathbb{Q}$.
   (c) Compute the degree $[K : \mathbb{Q}]$, where K is the splitting field of $f(x)$ over $\mathbb{Q}$.
   (d) How many intermediate fields are there between $\mathbb{Q}$ and K? Explain.
   (e) Prove that $K \subset \mathbb{R}$.

2. Let
$$A = \begin{bmatrix} -1 & 0 & 0 \\ -3 & 1 & 1 \\ 6 & -4 & -3 \end{bmatrix}.$$
   (a) Find the Jordan canonical form J of A.
   (b) Find an invertible matrix P such that $P^{-1}AP = J$. (You should not need to compute P^{-1}.)

3. (a) State the three Sylow theorems.
   (b) Prove that there is no simple group of order 108.

4. Prove that every group of order 45 is abelian. How many (nonisomorphic) groups of order 45 are there? Write down exactly one group from each isomorphism class.

5. Let $\sigma \in S_n$ where $n \geq 2$ and $\sigma \neq$ identity. Prove that it is possible to write σ as a product of $n-1$ or fewer transpositions. Moreover, if σ is not a n-cycle, then σ can be written as a product of $n-2$ or fewer transpositions.

6. A ring R (with 1) is called **simple** if its only (2-sided) ideals are (0) and R. The **center** of R is the subset
$$Z = z \in R | zr = rz \forall r \in R.$$
   Prove that the center of a simple ring is a field.

7. (a) Define "algebraic closure" of a field.
   (b) Prove that every field has an algebraic closure.

8. Let F be a field of prime characteristic p.

   (a) Prove that the map $\phi : F \to F, \phi(\alpha) = \alpha^p$, is a (ring) homomorphism.
   F is called **perfect** if ϕ is surjective.

   (b) Show that every finite field is perfect.

   (c) If F is any field of characteristic p, and x is an indeterminate, prove that $F(x)$ is **not** perfect.

9. The **affine group** $\mathcal{A}_n$ is the group of motions of $\mathbb{R}^n$ generated by $GL_n(\mathbb{R})$ together with the group $\mathcal{T}_n$ of translations: $t_a(x) = x + a\,(a, x \in \mathbb{R}^n)$. Prove that $\mathcal{T}_n$ is a normal subgroup of $\mathcal{A}_n$, and that $\mathcal{A}_n/\mathcal{T}_n \simeq GL_n(\mathbb{R})$.