Algebra Preliminary Examination

September 1993

1. Let G be a group that contains a subgroup H of index n. Show that G contains a normal subgroup K lying in H such that G/K is a finite group of order dividing $n!$.

2. Let S_4 denote the permutation group on 4 letters.
   (a) How many non-isomorphic 2-Sylow subgroups does S_4 have? Justify your answer.
   (b) How many 2-Sylow subgroups are there in each isomorphism class of 2-Sylow subgroups of S_4. Justify your answer

3. Describe the Galois group of the splitting field over $\mathbb{Q}$ of the polynomial $x^6 + 1$.

4. Give the definition of a *separable* field extension. Give an example of a finite field extension F/K that is not separable. Justify your answer.

5. In the ring $\mathbb{Z}[x]$, consider the ideal I generated by the elements 2 and $x^2 + x + 1$.
   (a) How many elements does the quotient ring $\mathbb{Z}[x]/I$ contain? Justify your answer.
   (b) Is the ideal I a maximal ideal? Justify your answer.

6. (a) Give the definition of a *projective* module. Give an example of a commutative ring A and an A-module M ($M \neq 0$) that is not projective. Justify your answer.
   (b) Give the definition of a *injective* module. Give an example of a commutative ring A and an A-module M ($M \neq 0$) that is not injective. Justify your answer.

7. Let A be a local domain with maximal ideal $\mathcal{M}$. (A is also assumed to be commutative with the identity element.)
   (a) State Nakayama's lemma for A.
   (b) Suppose now that $\mathcal{M}$ is a principal ideal. Show that if A is noetherian, then

$$\bigcap_{i=1}^{\infty} \mathcal{M}^i = (0).$$

8. (a) Let V be a non-zero vector space over $\mathbb{C}$, with a positive definite hermitian form $\langle\, ;\, \rangle : V \times V \to \mathbb{C}$. Let $A : V \to V$ be a hermitian map. Show that V has a orthogonal basis consisting of the eigenvectors of A.
   (b) Let $A \in M_n(\mathbb{C})$ be a hermitian matrix. Does there exist a matrix $B \in M_n(\mathbb{C})$ such that $B^n = A$? Justify your answer.