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Ph.D. Preliminary Examination Fall 1991

PROBABILITY THEORY

1. Give an example of two random variables (r.v.'s) X and Y, defined on a suitable probability space $(\Omega, \mathcal{F}, P)$, such that $E[X|Y] \neq E[E(X|Y)|X]$.

2. Let $\{F_n\}$ be a sequence of Gaussian distribution functions with means $\{m_n\}$ and variances $\{\sigma_n^2\}$. Show that $\{F_n\}$ is a tight sequence if and only if the sequences $\{m_n\}$ and $\{\sigma_n^2\}$ are bounded.

3. Let $(\Omega, \mathcal{F}, P)$ be the Probability space with $\Omega = [0, 1]$, $\mathcal{F}$ = the Borel σ-algebra on $[0, 1]$, and P = Lebesgue measure on $\mathcal{F}$. Define, for $n \geq 3$, the r.v.'s $X_n(\omega) = \frac{n}{\log n} I_{(0, n^{-1})}(\omega)$. Show that X_n, $n \geq 3$, are uniformly integrable and $E[X_n] \longrightarrow 0$ as $n \longrightarrow \infty$.

4. Let $k \geq 3$ be a prime number. Let X and Y be independent r.v.'s that are uniformly distributed on $\{0, 1, \ldots, k-1\}$. For $0 \leq n < k$, define $Z_n = X + nY$. Show that $Z_0, Z_1, \ldots, Z_{k-1}$ are pairwise independent and that if we know the values of two variables, then we know the values of all the variables.

5. Let $\{X_n, n \geq 1\}$ be a sequence of independent r.v.'s with $EX_n = 0$, $n \geq 1$. If $\sum_n \sigma^2(X_n) < \infty$, show that $\sum_n X_n$ converges almost surely.

6. Let $\{\mathcal{F}_n\}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$ and $\mathcal{F}_\infty := \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)$. For an $X \in L^1(\Omega)$, define $X_n := E[X|\mathcal{F}_n]$, $n \geq 1$. Show that $X_n \longrightarrow E(X|\mathcal{F}_\infty)$ almost surely and in $L^1(\Omega)$.

7. a) For a triangular array of r.v.'s, (i) define the uniform asymptotic negligility and (ii) state the Lindeberg condition.

   b) Let $\{X_n, n \geq 1\}$ be a sequence of independent normal r.v.'s with $EX_n = 0$, $n \geq 1$, $\sigma^2(X_1) = 1$, and $\sigma^2(X_n) = 2^{n-2}$, $n \geq 2$. Show that $\{X_n\}$ is neither uniformly asymptotically neglible nor satisfies the Lindeberg condition.