Qualifying Examination in Complex Analysis January 2012

All problems are equally weighted. For $a \in \mathbb{C}$ and $r > 0$, $B(a,r)$ denotes the open disk centered at a with radius r and $\bar{B}(a,r)$ denotes the closure of $B(a,r)$.

1. (a) State Cauchy's Integral Formula for functions holomorphic on $\bar{B}(a,r)$.
   (b) Use part (a) to prove that every holomorphic function on $B(a,r)$ can be represented by a power series.
   (c) Use part (a) to prove that every bounded entire function is constant.

2. Use the methods of complex analysis to evaluate $\displaystyle\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx$.

3. Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

4. Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic function. Show that there are $a, b \in \mathbb{C}$ such that $f(z) = az + b$. (Hint: Start by proving that f is a polynomial.)

5. Let $f(z) = \frac{z+2012}{5z^2+5z}$. Give the Laurent expansion of f that converges on
   (a) $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$.
   (b) $\{z \in \mathbb{C} \mid |z+1| > 1\}$.

6. Find a conformal map from $D = \{z : \ |z| < 1, \ |z - 1/2| > 1/2\}$ to the unit disk $B(0,1)$.