Complex

PhD. prelim in Complex Analysis, Spring 1997.

1. Prove any form of the Cauchy-Goursat theorem; (eg. a function holomorphic in an open set in $\mathbb{C}$ which contains the boundary $\gamma = \partial\Delta$ and interior of a triangle Δ, has zero integral around γ).

2. Prove the following form of the "Casorati-Weierstrass" theorem: if $f:\mathbb{C}\to\mathbb{C}$ is a holomorphic entire function, and if there is a neighborhood $U = \{z: |z| > r > 0\}$ of infinity whose image f(U) is not dense in $\mathbb{C}$, then f is a polynomial.

3. Prove that a sequence of functions holomorphic in an open set U, and which converges uniformly on all closed discs in U, has a limit which is holomorphic in U.

4. Use residues to calculate the real integral $\int_{-\infty}^{\infty}\frac{dx}{1+x^4}$, and justify your calculation.

5. **a)** Let $A = \{ z: \mathrm{Im}(z) > 0 \text{ and } -\pi/2 < \mathrm{Re}(z) < \pi/2 \}$. Find an explicit formula for a (one to one) conformal mapping of the region A onto the interior of the unit circle.
b) Let $B = \{ z: \mathrm{Re}(z) > (\mathrm{Im}(z))^2 - 1 \}$ be the region on the "right side" of the parabola $x = y^2 - 1$. <u>Prove</u> or <u>disprove</u>: for every pair of distinct points α,β in B, there is a (one to one) conformal mapping of the region B to itself, taking 0 to α, and 1 to β.

6. **(a)** Let $\Omega = \{n + mi$, for all integers n,m$\}$ denote the lattice of "Gaussian integers". Let f be a complex valued function holomorphic at all z *not* belonging to Ω, and assume that $f(z) = f(z+\omega)$ for all z in $\mathbb{C}-\Omega$, and all ω in Ω. If at $z = 0$, f is either holomorphic or has at worst a simple pole, prove f is constant.
(b) Construct a non constant meromorphic function g on $\mathbb{C}$, such that $g(z) = g(z+\omega)$ for all z in $\mathbb{C}-\Omega$, and all ω in Ω.

7. Classify all holomorphic automorphisms $f:\mathbb{C}\cup\{\infty\}\to\mathbb{C}\cup\{\infty\}$ of the Riemann sphere.

8. Construct an entire function with simple zeroes at the (positive) square roots of the positive integers, $\{n^{1/2}\}$, $n > 0$, and no other