Complex Analysis Qualifying Exam, Fall 2012

This version of the exam includes a corrected version of problem 6: the sentence in italics was omitted from the original version.

Instructions: Work all 7 problems. Problem 1 is worth 10 points and all the others are worth 15 points. (3 hour exam, 100 points total)

1. For each of these 2 kinds of mappings, state a version of the inverse function theorem.

   a) for a differentiable real mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$, and

   b) for a complex-differentiable mapping $f : \mathbb{C} \to \mathbb{C}$.

2. Give the Laurent series expansions of $\dfrac{z+1}{z(z-1)}$

   a) about $z = 0$, and

   b) about $z = 1$.

3. Compute the integral $\int_0^\infty \dfrac{dx}{(1+x^2)(1+9x^2)}$.

4. a) Let $\Omega \subset \mathbb{C}$ be an open set and suppose that $f : \Omega \to \mathbb{C}$ is an analytic function with $f'(z) \neq 0$ for all $z \in \Omega$. Show that f preserves angles. Include a definition of what angle-preserving means.

   b) Find a conformal map from the vertical strip $\{z : 0 < \text{Re } z < \pi\}$ onto the upper half-disk $\mathbb{D} \cap \{\text{Im } z > 0\}$.

5. Assuming that $|b| < 1$, show that $f(z) = z^3 + 3z^2 + bz + b^2$ has exactly 2 roots (counting multiplicity) in $|z| < 1$.

6. Suppose that $\{f_n\}$ is a sequence of analytic functions on an open set $\Omega \subset \mathbb{C}$ converging pointwise to a function f on Ω. *Suppose in addition that the sequence is locally uniformly bounded, i.e. for each compact subset $K \subset \Omega$ there is a constant $C < \infty$ such that $|f_n(z)| \leq C$ for all $z \in K$ and all n.* Prove that f is an analytic function on Ω and that the convergence is uniform on compact subsets of Ω. Give an example to illustrate that the convergence need not be uniform on all of Ω.

7. Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk. Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic, and admits a continuous extension $\bar{f} : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ such that $|\bar{f}(z)| = 1$ whenever $|z| = 1$.

   a) Prove that f is a rational function.

   b) Suppose that $z = 0$ is the unique solution of $f(z) = 0$. Prove that $f(z) = \lambda z^n$ for some $\lambda \in \mathbb{C}, |\lambda| = 1$, and $n \in \mathbb{N}$.

   c) More generally, suppose that $a_1, \ldots, a_n \in \mathbb{D}$ are the zeroes of f, listed with multiplicity. Prove that

$$f(z) = \lambda \prod_{i=1}^{n} \frac{z - a_i}{1 - \bar{a}_i z}, \quad |\lambda| = 1.$$

(Hint: $z \mapsto \frac{z-a_i}{1-\bar{a}_i z}$ is an automorphism of $\mathbb{D}$ taking $a_i \mapsto 0$.)