QUALIFYING EXAMINATION IN COMPLEX ANALYSIS

August 12, 2011, 9–11 a.m.

$\mathbb{D}$ denotes the (open) unit disk, $\overline{\mathbb{D}}$ the closed unit disk, and $\mathbb{T} = \partial\mathbb{D}$ the unit circle. Provide justifications as appropriate.

1. (15 points) Use methods of complex analysis to evaluate
$$\int_0^\infty \frac{\sqrt{x}}{(x+1)^2}\,dx$$
Be sure to provide complete justifications.

2. (15 points) Let c be a complex number such that $|c| < 1/3$. Show that on the open set $\mathcal{R}e(z) < 1$ the function $f(z) = ce^z$ has exactly one fixed point, i.e., a point z_0 such that $f(z_0) = z_0$.

3. (20 points) Let $\overline{B}(a,r)$ denote the closed disk of radius $r > 0$ about a point $a \in \mathbb{C}$. Let f be a holomorphic function on an open set containing $\overline{B}(a,r)$, and let $M = \sup\limits_{z\in\overline{B}(a,r)} |f(z)|$. Prove that for $z \in \overline{B}(a,r/2), z \neq a$, we have
$$\frac{|f(z)-f(a)|}{|z-a|} \leq \frac{2M}{r}.$$

4. (15 points) Suppose $\Omega \subset \mathbb{C}$ is a region containing $\overline{\mathbb{D}}$ and f is holomorphic on Ω. Suppose that on $\mathbb{D}$ we have $f(z) = \sum a_n z^n$ and the series has radius of convergence equal to 1.
   a. Give an example of such an f so that the series converges at every point of $\mathbb{T}$.
   b. Give an example of such an f that is analytic at $z_0 \in \mathbb{T}$ and for which $\sum a_n z_0^n$ diverges.
   c. Prove that f cannot be analytic at *every* point of $\mathbb{T}$.

5. (20 points) Consider $f(x,y) = x^2 - 2y + y^3$. Let $P = (1,1)$.
   a. Let $X = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\}$. Prove that there is a neighborhood U of P so that $U \cap X$ is given by $y = \phi(x)$ for some $\mathcal{C}^1$ function ϕ. Give $\phi'(x)$.
   b. Now consider the same equation in $\mathbb{C}^2$ (i.e., $Y = \{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\}$). Prove that the analogous statement holds. What is $\phi'(x)$ as an $\mathbb{R}$-linear map from $\mathbb{C}$ to $\mathbb{C}$? Why is ϕ holomorphic? (Hint: If you consider $g\colon \mathbb{C} \to \mathbb{C}$ as a map $\tilde{g}\colon \mathbb{R}^2 \to \mathbb{R}^2$, how is holomorphicity of g characterized by $D\tilde{g}$?)

6. (15 points) Do **either** a **or** b.
   a. Suppose f is meromorphic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ except at finitely many points of $\mathbb{D}$, and real on $\mathbb{T}$. Prove that f is a rational function.
   b. Let $\Omega \subset \mathbb{C}$ be the region inside the unit circle $|z| = 1$ and outside the circle $|z - \frac{1}{4}| = \frac{1}{4}$. Find a one-to-one conformal map from Ω onto an annulus $r < |z| < 1$ for the appropriate value of r.