COMPLEX ANALYSIS PRELIM - SEPTEMBER 1997

Definition 1. Below, $B(a,r) = \{z \in \mathbb{C} \mid |z-r| < a\}$, $\bar{B}(a,r) = \{z \in \mathbb{C} \mid |z-r| \leq a\}$, $\partial B(a,r) = \{z \in \mathbb{C} \mid |z-r| = a\}$.

Problem 1. Find all possible values of the integral

$$\int_\gamma \frac{\sin z\, dz}{z^3 + z^4},$$

where γ is a closed smooth curve.

Problem 2. Let $a, b \in \mathbb{C}$ be two points and γ be a smooth curve joining them, and let $g(\xi)$ be a continuous function on γ. Prove that the function $f(z)$ defined by the formula

$$f(z) = \int_\gamma \frac{g(\xi)d\xi}{\xi - z}$$

is analytic on $\mathbb{C} \setminus \gamma$. Justify every step.

Problem 3. Which of the following do or do not exist? Prove your answers using the standard theorems if you need to.

1. An entire function whose zeros are $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$.
2. A meromorphic function on $\mathbb{C}$ whose poles are $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$.
3. A meromorphic function on $\mathbb{C}$ whose poles are $\{1/n \mid n \in \mathbb{N}\}$.
4. A meromorphic function on $\mathbb{C} \setminus \{0\}$ whose poles are $\{1/n \mid n \in \mathbb{N}\}$ with residues n.

Problem 4. Classify all holomorphic automorphisms of $\mathbf{C} \setminus \{0\}$.

Problem 5. Let $f(z)$ be a function analytic on the upper half-plane $Im\, z \geq 0$ which has a non-essential singularity at ∞, and assume that $|f(z)| = 1$ when $Im\, z = 0$. Find a general formula for $f(z)$. Hint: start with the case when f has no zeros.

Problem 6. How many zeros does the function $f(z) = z^4 - 5z + 1$ have in $B(0,1)$? Write down the first two nonconstant terms in the power expansion about $\lambda = 1$ of a root $z(\lambda)$ of the equation $z^4 - 5z + \lambda = 0$.

Problem 7. Let $f(z)$ be a double-periodic meromorphic function with the periods $\Lambda = \{n + mi \mid n, m \in \mathbb{N}\}$.

1. Prove that f cannot have exactly one zero of multiplicity 1 (and no other zeros) modulo the periods.
2. Give an example of such a function that has exactly one zero of multiplicity 3.
3. Describe an explicit embedding of $\mathbb{C}/\Lambda$ minus a point into $\mathbb{C}^2$.

Problem 8. Let $f(z)$ be an analytic function on $B(0,1)$. Assume that $|f(z)| = \phi(x)\psi(y)$ for all $z = x + iy \in B(0,1)$ and some functions ϕ, ψ. Prove that either $f(z) \equiv 0$ or $f(z) = \exp(az^2 + bz + c)$ for some $a \in \mathbb{R}$ and $b, c \in \mathbb{C}$.

Problem 9. Prove that there exists a holomorphic function f on $B(0,1)$ which cannot be analytically extended to any larger connected open region. Next, prove the same for an open square $\{|x| < 1, |y| < 1\}$. Can you generalize?