Complex Analysis Qualifying Exam, Spring 2016

The problems count equally. Give clear reasoning and state clearly which theorems you are using.

1. Use residues to evaluate the definite integral $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx$. (Include justification.)

2. Describe all conformal analytic maps of the quarter disk

$$Q := \{z \in \mathbb{C} : |z| < 1, \operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0\}$$

onto itself. Your answer may involve composites, but justify that your list is complete.

3. State and prove the Weierstrass M-test for series of complex-valued functions.

4. Find, with justification, all continuous maps f from the closed half-plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\}$ to $\mathbb{C}$ which satisfy

   i) $f(\mathbb{R}) \subset \mathbb{R}$,

   ii) f is analytic on the interior of $\mathbb{H}$, and

   iii) $\lim_{z \to \infty} f(z) = \infty$.

5. Support your answers to each part of this problem with appropriate proofs and examples.

   a) Suppose f, g have poles at the origin. What can be said about the singularity of their product $f \cdot g$ at the origin?

   b) Repeat part a) when f, g have essential singularities at the origin.

   c) Repeat part a) when f has a pole at the origin while g has an essential singularity at the origin.

6. Suppose f is entire and $|f(z^2)| \leq |f(z)|$ for all $z \in \mathbb{C}$. Prove that f is constant.

7. Let (f_n) be a sequence of functions which are analytic on a domain G and which converge uniformly to a function g on G. Show that if g has an isolated zero in G, then for all sufficiently large n, f_n also has a zero in G.