QUALIFYING EXAMINATION IN COMPLEX ANALYSIS

January 4, 2008, 2–4 pm

As usual, $\mathbb{D}$ denotes the (open) unit disk and $\mathbb{P}^1$ denotes the Riemann sphere, $\mathbb{C} \cup \{\infty\}$. Provide justifications as appropriate.

1. (20 points) Use methods of complex analysis to evaluate

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\, dx\,.$$

2. (15 points) Let $\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and $\Omega' = \{z \in \mathbb{C} : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}$. Give a conformal bijective mapping from Ω to Ω'.

3. (25 points) Suppose f_n are holomorphic functions on a domain Ω. Prove that if $f_n \to f$ uniformly on compact subsets of Ω, then
   a. f is holomorphic;
   b. $f_n' \to f'$ uniformly on compact subsets of Ω;
   c. if each f_n is one-to-one, then f is either constant or one-to-one.

4. (20 points)
   a. Suppose $f = u + iv$ is holomorphic at a. Explain carefully the relation between the complex derivative $f'(a)$ and the Jacobian matrix at a of f viewed as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ (denoted $J_f(a)$). In particular, how is $\det J_f(a)$ related to $f'(a)$?
   b. Consider $X = \{(z, w) \in \mathbb{C}^2 : z^4 + 4zw^3 + w^4 = 1\}$. Prove that for each $(z_0, w_0) \in X$ there are neighborhoods $U \subset \mathbb{C}$ and $V \subset \mathbb{C}$ of z_0 and w_0 respectively so that $X \cap (U \times V)$ can be represented either as a graph $w = g(z)$ for some holomorphic function g defined on U or as a graph $z = h(w)$ for some holomorphic function h defined on V. (Hint: Start with the implicit function theorem and a map from $\mathbb{R}^4$ to $\mathbb{R}^2$.)

5. (20 points) Do **either** part a. **or** part b.
   a. What are all the bijective holomorphic mappings from $\mathbb{P}^1$ to $\mathbb{P}^1$? Prove your claim, stating clearly any results that you use.
   b. Suppose $f\colon \mathbb{D} \to \mathbb{D}$ is holomorphic, 0 is a zero of order k, $f(z) \neq 0$ whenever $z \neq 0$, and $\lim\limits_{|z| \to 1} |f(z)| = 1$. Give, with proof, a formula for $f(z)$.