Algebra qualifying exam, Spring 2019

Directions: Justify all the calculations and state the theorems you use in your answers. Each problem is worth 10 points. In the solution of a part of a problem, you may use any earlier part of that problem, whether or not you've correctly solved it.

1. Let A be a square matrix over the complex numbers. Suppose that A is nonsingular and that A^{2019} is diagonalizable over $\mathbb{C}$. Show that A is also diagonalizable over $\mathbb{C}$.

2. Let $F = \mathbb{F}_p$, where p is a prime number.

   (a) Show that if $\pi(x) \in F[x]$ is irreducible of degree d, then $\pi(x)$ divides $x^{p^d} - x$.

   (b) Show that if $\pi(x) \in F[x]$ is an irreducible polynomial that divides $x^{p^n} - x$, then $\deg \pi(x)$ divides n.

3. How many isomorphism classes are there of groups of order 45? Describe a representative from each class.

4. For a finite group G, let $c(G)$ denote the number of conjugacy classes of G.

   (a) Prove that if two elements of G are chosen uniformly at random, then the probability they commute is precisely $c(G)/|G|$.

   (b) State the *class equation* for a finite group.

   (c) Using the class equation (or otherwise) show that the probability in part (a) is at most $\frac{1}{2} + \frac{1}{2 \cdot [G : Z(G)]}$. Here, as usual, $Z(G)$ denotes the center of G.

5. Let R be an integral domain. Recall that if M is an R-module, the *rank* of M is defined to be the maximum number of R-linearly independent elements of M.

   (a) Prove that for any R-module M, the rank of $\text{Tor}(M)$ is 0.

   (b) Prove that the rank of M = rank of $M/\text{Tor}(M)$.

   (c) Suppose that M is a non-principal ideal of R. Prove that M is torsion-free of rank 1 but not free.

6. Let R be a commutative ring with 1.

   (a) Show that every proper ideal of R is contained within a maximal ideal.

   (b) Let $J(R)$ denote the intersection of all maximal ideals of R. Show that

   $$x \in J(R) \iff 1 + rx \text{ is a unit for all } r \in R.$$

   (c) Suppose now that R is finite. Show that in this case $J(R)$ consists precisely of the nilpotent elements in R. (Recall that $x \in R$ is *nilpotent* if $x^n = 0$ for some positive integer n.)

7. Let p be a prime number. Let A be a $p \times p$ matrix over a field F, with 1 in all entries except 0 on the main diagonal. Determine the Jordan canonical form (JCF) of A

   (a) when $F = \mathbb{Q}$,

   (b) when $F = \mathbb{F}_p$.

(Hint: In both cases, all eigenvalues lie in the ground field.) In each case find a matrix P such that $P^{-1}AP$ is in JCF.

8. Let $\zeta = e^{2\pi i/8}$.

   (a) What is the degree of $\mathbb{Q}(\zeta)/\mathbb{Q}$?

   (b) How many quadratic subfields of $\mathbb{Q}(\zeta)$ are there?

   (c) What is the degree of $\mathbb{Q}(\zeta, \sqrt[4]{2})$ over $\mathbb{Q}$?