Algebra Qualifying Examination
Spring 2016

Justify all the calculations and state the theorems you use in your answers. Each problem is worth 10 points.

1. Let

$$A = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix} \in M_3(\mathbb{C})$$

(a) Find the Jordan canonical form J of A.
(b) Find an invertible matrix P such that $P^{-1}AP = J$. (You do not need to compute P^{-1}.)

2. Let $K = \mathbb{Q}[\sqrt{2} + \sqrt{5}]$.
(a) Find $[K : \mathbb{Q}]$.
(b) Show that $K/\mathbb{Q}$ is Galois, and find the Galois group G of $K/\mathbb{Q}$.
(c) Exhibit explicitly the correspondence between subgroups of G and intermediate fields between $\mathbb{Q}$ and K.

3. (a) State the three Sylow theorems.
(b) Prove that any group of order 1225 is abelian.
(c) Write down exactly one representative in each isomorphism class of (abelian) groups of order 1225.

4. Let R be a ring with the following commutative diagram of R-modules (each row represents a short exact sequence of R-modules).

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}$$

Prove that if α and γ are isomorphisms then β is an isomorphism.

5. Let G be a finite group acting on a set X. For $x \in X$, let G_x be the stabilizer of x, and $G \cdot x$ be the orbit of x.

(a) Prove that there is a bijection between the left cosets G/G_x and $G \cdot x$.

(b) Prove that the center of every finite p-group G is non-trivial by considering the action of G on $X = G$ by conjugation.

6. Let K be a Galois extension of a field F with $[K : F] = 2015$. Prove that K is an extension by radicals of the field F.

7. Let $D = \mathbb{Q}[x]$ where $\mathbb{Q}$ are the rational numbers and let M be a $\mathbb{Q}[x]$-module such that

$$M \cong \frac{\mathbb{Q}[x]}{(x-1)^3} \oplus \frac{\mathbb{Q}[x]}{(x^2+1)^3} \oplus \frac{\mathbb{Q}[x]}{(x-1)(x^2+1)^5} \oplus \frac{\mathbb{Q}[x]}{(x+2)(x^2+1)^2}.$$

Determine the elementary divisors and invariant factors of M.

8. Let R be a simple rng (a nonzero ring which is not assumed to have a 1, whose only two-sided ideals are $\{0\}$ and R) satisfying the following two conditions:
(i) R has no zero-divisors; and
(ii) if $0 \neq x \in R$ then $2x \neq 0$ (where $2x := x + x$).

Prove the following:
(a) For each $x \in R$ there is one and only one element $y \in R$ such that $x = 2y$.
(b) Suppose $x, y \in R$ such that $x \neq 0$ and $2(xy) = x$. Then $yz = zy$ for all $z \in R$.

[You can get partial credit for part (b) if you do it assuming R has a 1.]