Directions: Justify all the calculations and state the theorems you use in your answers. In multi-part problems, you may assume the result of any part (even if you have not been able to do it) in working on subsequent parts.

1. (10 points) Suppose the group G acts on the set A. Assume this action is faithful (recall that this means that the kernel of the homomorphism from G to $\mathrm{Sym}(A)$ which gives the action is trivial) and transitive (for all a, b in A, there exists g in G such that $g \cdot a = b$.)

   (a) For $a \in A$, let G_a denote the stabilizer of a in G. Prove that for any $a \in A, \cap_{\sigma \in G} \sigma G_a \sigma^{-1} = 1$.

   (b) Suppose that G is abelian. Prove that $|G| = |A|$. Deduce that every abelian transitive subgroup of S_n has order n.

2. (15 points)

   (a) Classify the abelian groups of order 36.

   For the rest of the problem, assume that G is a non-abelian group of order 36. (You may assume that the only subgroup of order 12 in S_4 is A_4 and that A_4 has no subgroup of order 6.)

   (b) Prove that if the 2-Sylow subgroup of G is normal, G has a normal subgroup N such that G/N is isomorphic to A_4.

   (c) Show that if G has a normal subgroup N such that G/N is isomorphic to A_4 and a subgroup H isomorphic to A_4 it must be the direct product of N and H.

   (d) Show that the dihedral group of order 36 is a non-abelian group of order 36 whose Sylow-2 subgroup is not normal.

3. (10 points) Let F be a field. Let $f(x)$ be an irreducible polynomial in $F[x]$ of degree n and let $g(x)$ be any polynomial in $F[x]$. Let $p(x)$ be an irreducible factor (of degree m) of the polynomial $f(g(x))$. Prove that n divides m. Use this to prove that if r is an integer which is not a perfect square, and n is a positive integer then every irreducible factor of $x^{2n} - r$ over $\mathbb{Q}[x]$ has even degree.

4. (10 points)

   (a) Let $f(x)$ be an irreducible polynomial of degree 4 in $\mathbb{Q}[x]$ whose splitting field K over $\mathbb{Q}$ has Galois group $G = S_4$. Let θ be a root of $f(x)$. Prove that $\mathbb{Q}[\theta]$ is an extension of $\mathbb{Q}$ of degree 4 and that there are no intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}[\theta]$.

   (b) Prove that if K is a Galois extension of $\mathbb{Q}$ of degree 4, then there is an intermediate subfield between K and $\mathbb{Q}$.

5. (10 points) A ring R is called *simple* if its only two-sided ideals are 0 and R.

   (a) Suppose R is a commutative ring with 1. Prove R is simple if and only if R is a field.

   (b) Let k be a field. Show the ring $M_n(k)$, $n \times n$ matrices with entries in k, is a simple ring.

6. (15 points) For a ring R, let $U(R)$ denote the multiplicative group of units in R. Recall that in an integral domain R, $r \in R$ is called *irreducible* if r is not a unit in R, and the only divisors of r have the form ru with u a unit in R. We call a non-zero, non-unit $r \in R$ *prime* in R if $r|ab$ implies $r|a$ or $r|b$. Consider the ring $R = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}\}$.

   (a) Prove R is an integral domain.

   (b) Show $U(R) = \{\pm 1\}$.

   (c) Show 3, $2 + \sqrt{-5}$, and $2 - \sqrt{-5}$ are irreducible in R.

   (d) Show 3 is not prime in R.

   (e) Conclude R is not a PID.

7. (15 points) Let F be a field and let V and W be vector spaces over F. Make V and W into $F[X]$-modules via linear operators T on V and S on W by defining $X \cdot v = T(v)$ for all $v \in V$ and $X \cdot w = S(w)$ for all $w \in W$. Denote the resulting $F[X]$-modules by V_T and W_S respectively.
   (a) Show that an $F[X]$-module homomorphism from V_T to W_S consists of an F-linear transformation $R : V \to W$ such that $RT = SR$.

   (b) Show that $V_T \cong W_S$ as $F[X]$-modules if and only if there is an F-linear isomorphism $P : V \to W$ such that $T = P^{-1}SP$.

   (c) Recall that a module M is *simple* if $M \neq 0$ and any proper submodule of M must be zero. Suppose that V has dimension 2. Give an example of F, T with V_T simple.

   (d) Assume F is algebraically closed. Prove that if V has dimension 2, then any V_T is not simple.