ALGEBRA QUALIFYING EXAM SPRING 2024

Each problem is worth 10 points.

(1) Let q be a prime integer and let $G = S_q$, the symmetric group on q elements. Let $a = (12\cdots q)$ be a $q-$cycle in G.
(a)Find the order of $C_G(a)$, the centralizer of a in G.
(b)Let $Q = \langle a \rangle$, the cyclic subgroup of G generated by a. Find the order of $N_G(Q)$.

(2) Let r be a rational number which is not a perfect square. Let n be a positive integer.Let $g(x) \in \mathbb{Q}[x]$ be an irreducible factor of the polynomial $f(x) = x^{2n} - r$ in $\mathbb{Q}[x]$. Prove that $g(x)$ has even degree..

(3) Use the Fundamental Theorem for finitely generated abelian groups to prove that the multiplicative group F^* of a finite field F is cyclic.

(4) Let R be an integral domain.
(a)Prove that if $a \in R$ is prime then a is irreducible in R.
(b)Prove that if R is a Unique Factorization Domain then every irreducible element of R is prime.
(c) Give an example of a ring R and an element $a \in R$ which is irreducible but not prime.

(5) Let A be the $n \times n$ matrix over $\mathbb{Q}$ with 2's on the diagonal , 1's just below the diagonal, and 0's everywhere else, i.e. $A = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}$.(For example, if $n = 4, A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$.) Find J, the Jordan Canonical Form of A.
(b)Find the minimal polynomial of A. Justify your answer.

(6) (a) Let E be an extension field of F with $[E : F] = n$ and let $\alpha \in E$. Prove that α belongs to no proper subfield of E if and only if the minimal polynomial of α/F has degree n.
(b)Let q be a prime integer. Find the number of monic irreducible polynomials of degree q in $\mathbb{F}_p[x]$.
(c)Let q be a prime integer. Find the number of monic irreducible polynomials of degree q^2 in $\mathbb{F}_p[x]$.
(d) Let q, r be distinct prime integers. Find the number of monic irreducible polynomials of degree qr in $\mathbb{F}_p[x]$.

(7))Let q be a prime integer and let $f(x) = x^q - 2$ in $\mathbb{Q}[x]$.Let ω be a primitive q-th root of unity.
(a)Prove that the splitting field of $f(x)$ is $\mathbb{Q}(2^{1/q}, \omega)$.
(b)Let G be the Galois group of $f(x)/\mathbb{Q}$. Prove that for each $b \in \mathbb{Z}_q$ and each $a \in \mathbb{Z}_q^*$ there is a unique element $\sigma_{a,b}$ of G which sends $2^{1/q} \to 2^{1/q}\omega^b$ and $\omega \to \omega^a$. You may assume that $f(x)$ is irreducible in $\mathbb{Q}(\omega)[x]$.
(c)Let G_1 be the subgroup of $GL_2(\mathbb{F}_q)$ consisting of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $b \in \mathbb{Z}_q$ and $a \in \mathbb{Z}_q^*$. Prove that the map $\tau : G \to G_1$ given by $\sigma_{a,b} \to \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ is an isomorphism of groups.

(8) Let p, q be prime integers with $p < q$.

(a)Prove that every group G of order pq is a semidirect (perhaps direct) product of the cyclic groups C_p with C_q .

(b)Prove that if p does not divide $q - 1$ then G is cyclic.

(c)Prove that if p does divide $q - 1$ then there is a G which is not abelian.