UNIVERSITY OF GEORGIA

DEPARTMENT OF MATHEMATICS
MATH 2250 - FINAL EXAM
FALL 2024

PRINTED NAME : Answer Key
STUDENT ID : ____________
DATE : ___/___/___

GRADE ___ /150

INSTRUCTOR : ____________
CLASS TIME : ____________

INSTRUCTIONS

№	SCORE	MAX
1		5
2		5
3		5
4		5
5		5
6		5
7		5
8		5
9		5
10		5
11		5
12		5
13		5
14		5
15		5
16		5
17		10
18		5
19		5
20		5
21		10
22		15
23		10
24		10
TOTAL		150

- The exam lasts 3 hours and it has two parts: the first one consists of Multiple Choice (MC) questions, and the second part of Free Response (FR) ones. You must show work for both parts, unless explicitly told otherwise. An unjustified answer will receive no credit. If you are using a shortcut, explain it.
- Your work must be neat and organized. Circle the answer for MC questions and put a box around the final answer for the FR questions. There is only one correct choice for each MC question.
- Smart devices, including smart watches and cell phones, are prohibited and must not be within reach.
- If you plan to use a calculator, only TI-30XS MultiView (the name must match exactly) is permitted; no other calculators or sharing of calculators is allowed.
- Provide an exact answer for each problem. Answers containing symbolic expressions such as $\cos(3)$ and $\ln(2)$ are perfectly acceptable, but $\sin(\pi/2) = 1$.
- If additional space is needed, use the last two pages. Write "cont'd" (continued) in the designated area and continue on the scrap paper by first writing the problem number and then continuing your solution. Work outside the specified area without any indication, will not be graded.

Part I: Multiple Choice

Show work and circle the final answer.

_____ PTS 1. [5 pts] Find the derivative of $y = \ln(5 + \arcsin(3x))$.

(A) $y' = \dfrac{\sqrt{1-9x^2}}{2}$

(B) $y' = \dfrac{3}{(5+\arcsin(3x))\sqrt{1-9x^2}}$

(C) $y' = \dfrac{3}{5+\sqrt{1-9x^2}}$

(D) $y' = \dfrac{3}{5+\sqrt{9x^2-1}}$

(E) $y' = 3\left(5+\sqrt{9x^2-1}\right)$

$$y' = \frac{1}{5+\arcsin(3x)} \cdot (5+\arcsin(3x))'$$

$$= \frac{1}{5+\arcsin(3x)} \cdot 3 \cdot \frac{1}{\sqrt{1-9x^2}}$$

_____ PTS 2. [5 pts] In which of the following intervals is the function $f(x) = \frac{1}{2}x^4 - x^3 + 5x + 1$ concave down?

(A) $(-\infty, \infty)$

(B) $(-\infty, 0)$

(C) $(0, \frac{1}{2})$

(D) $(0, 1)$

(E) $(1, \infty)$

$$f'(x) = 2x^3 - 3x^2 + 5$$

$$f''(x) = 6x^2 - 6x$$

$$= 6x(x-1)$$

$f''(x) = 0$ at $x = 0$ and $x = 1$

x:		0		1	
sign of f''	+	0	−	0	+
			concave down		

$f(x)$ is concave down on $(0, 1)$.

_____ PTS 3. [5 pts] The graph of $y = \dfrac{x+2}{x^2-3x-10}$ has

(A) vertical asymptotes at $x = -2$ and $x = 5$.

(B) a vertical asymptote at $x = 2$ and a removable discontinuity at $x = -5$.

(C) a vertical asymptote at $x = -2$ and a removable discontinuity at $x = 5$.

(D) a vertical asymptote at $x = 5$ and a removable discontinuity at $x = -2$.

(E) removable discontinuities at $x = -2$ and $x = 5$.

$y = \dfrac{x+2}{(x+2)(x-5)} = \dfrac{1}{(x-5)}$ for $x \neq -2$.

The graph of y has a removable discontinuity at $x=-2$ and a vertical asymptote at $x = 5$ $\left(\lim_{x \to 5^{+,-}} y = \pm\infty\right)$.

_____ PTS 4. [5 pts] The largest interval over which the function $f(x) = \sqrt{12x+12} - x$ is increasing is $(-1, a)$. What is a?

(A) $-\frac{1}{2}$

(C) 1

(E) 2

(B) 0

(D) $\frac{3}{2}$

(F) $+\infty$

dom $(f(x)) = [-1, \infty)$

$f'(x) = \frac{1}{\cancel{2}} \frac{1}{\sqrt{12x+12}} \cdot \cancel{12}^{6} - 1$

$= \dfrac{6 - \sqrt{12x+12}}{\sqrt{12x+12}}$ $=0$: $6 - \sqrt{12x+12} = 0 \iff 36 = 12x+12 \iff x = 2$

DNE: $x = -1$

x:		-1		2	
sign of f'	//////	undefined	$+$	0	$-$
			increasing		

$f(x)$ is increasing on $(-1, 2)$ so $a = 2$.

.

______ PTS 5. [5 pts] Which of these lines is parallel to the line tangent to the graph of $f(x) = \ln(\pi) + xe^{2x}$ at the point $x = -1$?

(A) $y = -x + \pi$

(B) $y = \ln(\pi)$

(C) $y = -\frac{x}{e^2} + 2\ln(\pi)$

(D) $y = 3e^2x + \ln(\pi)$

(E) None of those.

$$f'(x)\Big|_{x=-1} = e^{2x}\Big|_{x=-1} + 2xe^{2x}\Big|_{x=-1}$$

$$= e^{-2} - 2e^{-2}$$

$$= -e^{-2}$$

Parallel lines are characterized by the same slope, thus the correct answer is C.

______ PTS 6. [5 pts] Consider the two functions defined below:

$$f(x) = \frac{1}{1+e^x} \qquad\qquad g(x) = \frac{5x^7 + 4x^3 + 9}{4x^7 - 1}.$$

Which of the following describes the number of horizontal asymptotes of the graph of each function?

(A) The graph of $f(x)$ contains **two horizontal asymptotes**, while the graph of $g(x)$ contains **one horizontal asymptote**.

(B) The graph of $f(x)$ contains **one horizontal asymptote**, while the graph of $g(x)$ contains **two horizontal asymptotes**.

(C) The graphs of both $f(x)$ and $g(x)$ contain exactly **one horizontal asymptote**.

(D) The graphs of both $f(x)$ and $g(x)$ contain exactly **zero horizontal asymptote**.

(E) None of the above are correct.

$\lim\limits_{x\to\infty} \frac{1}{1+e^x} = 0 \therefore y=0$ is a h.a. for $f(x)$.

$\lim\limits_{x\to-\infty} \frac{1}{1+e^x} = 1 \therefore y=1$ -"- -"- -"- $f(x)$.

$\lim\limits_{x\to\pm\infty} \frac{5x^7+4x^3+9}{4x^7-1} = \frac{5}{4} \therefore y=\frac{5}{4}$ is a h.a. for $g(x)$.

Thus $f(x)$ has two horizontal asymptotes and $g(x)$ just one.

______ PTS 7. [5 pts] Given the function $f(x)$ defined as

$$f(x) = \begin{cases} \ln(x+k), & \text{if } x \geq 0, \\ \frac{\sin(2x)}{x}, & \text{if } x < 0 \end{cases}$$

find the value of k such that $f(x)$ is continuous for all x.

- (A) 1
- (B) e
- (C) e^2
- (D) 2
- (E) $-\infty$

If $f(x)$ is continuous everywhere:

$$\lim_{x \to 0^+} \ln(x+k) = \lim_{x \to 0^-} \frac{\sin(2x)}{x}$$

$$\ln(k) = \lim_{2x \to 0^+} 2\,\underbrace{\frac{\sin(2x)}{2x}}_{=1}$$

$$\ln k = 2$$

$$\Rightarrow \quad k = e^2.$$

______ PTS 8. [5 pts] Find the following limit provided that $\lim_{x \to (-2)} f(x) = 0$ and $\lim_{x \to (-2)^-} f'(x) = 2$.

$$\lim_{x \to (-2)^-} \frac{\sin(\pi x)}{f(x)}$$

- (A) $\frac{1}{2}$
- (B) $\frac{\pi}{2}$
- (C) $-\frac{\pi}{2}$
- (D) None of the above, but the limit is well defined.
- (E) The limit does not exist.

$$\lim_{x \to (-2)^-} \frac{\sin(\pi x)}{f(x)} \qquad \text{form: } \frac{0}{0}$$

$$\stackrel{\text{by L'Hôpital}}{=} \lim_{x \to (-2)^-} \frac{\pi \cos(\pi x)}{f'(x)}$$

$$= \frac{\pi}{2}.$$

______ PTS 9. [5 pts] Let $f(x) = x^{4/3} - 24x^{1/3}$. Which of the following best describes the critical points of f? Show your work in computing and labeling the critical points of f.

- (A) The graph of $f(x)$ attains a local maximum at $x = 0$ and a local minimum at $x = 3$.
- (B) The graph of $f(x)$ attains a local minimum at $x = 0$ and a local maximum at $x = 3$.
- (C) The graph of $f(x)$ attains a local minimum at $x = 0$ and a local minimum at $x = 6$.
- (D) The graph of $f(x)$ attains a local maximum at $x = 0$ and a local minimum at $x = 6$.
- (E) None of the above are correct.

$\mathrm{dom}(f(x)) = (-\infty, \infty)$

$f'(x) = \frac{4}{3}x^{1/3} - 8x^{-2/3}$

$= \frac{4x - 24}{3x^{2/3}}$ $=0$: $x = 6$; DNE: $x = 0$

x:		0		6	
sign of f'	$-$	undefined	$-$	0	$+$
				local minimum	

$\therefore$ $f(x)$ attains a local minimum at $x = 6$.

______ PTS 10. [5 pts] Determine the absolute maximum and minimum values of $f(x) = \frac{\ln(x)}{x}$ on $[1, e^4]$.

- (A) This function does not attain a maximum value but has minimum value $\frac{1}{e}$.
- (B) This function has maximum value $\frac{1}{e}$ and minimum value of $\frac{4}{e^4}$.
- (C) This function has maximum value $\frac{1}{e}$ and minimum value 0.
- (D) This function has maximum value 1 and minimum value e.
- (E) This function does not attain a maximum or minimum value.

$\mathrm{dom}(f(x)) = (0, \infty)$

$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$ $=0$: $x = e$; DNE: $x = 0 \notin \mathrm{dom}(f(x))$

$f(1) = \frac{\ln(1)}{1} = 0$ ← absolute minimum

$f(e^4) = \frac{\ln(e^4)}{e^4} = \frac{4}{e^4}$

$f(e) = \frac{\ln(e)}{e} = \frac{1}{e}$ ← absolute maximum

______ PTS 11. [5 pts] A car's position along a straight road is modeled by the equation:

$$e^z + \int_3^z v(t)\,dt = z^3 + 5z$$

where $v(t)$ is the velocity of the car at time t. Find the value of $v(0)$.

- (A) 0
- (B) 1
- (C) 3
- (D) 4
- (E) 5

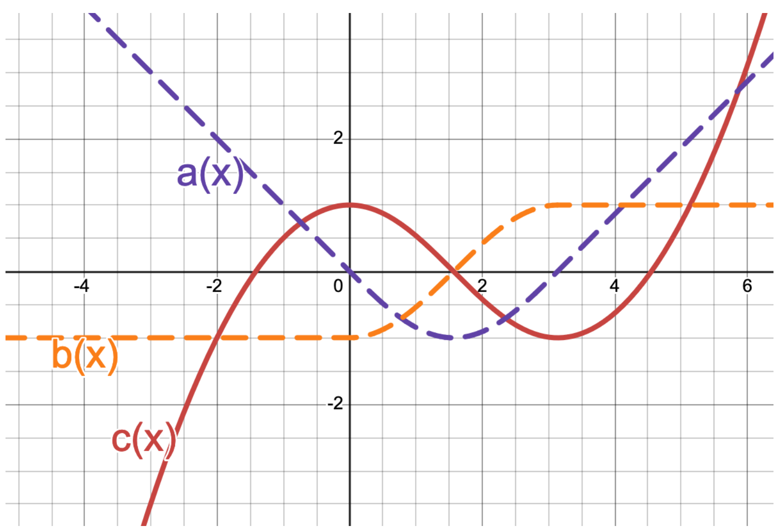
To obtain $v(t)$ we need to differentiate both sides; using the FTC:

$$e^z + v(z) = 3z^2 + 5$$

At $z = 0$:

$$1 + v(0) = 5 \Rightarrow v(0) = 4.$$

______ PTS 12. [5 pts] Lisa has plotted the graphs of f, f' and f''. Unfortunately, she accidentally lost track of what happens and labelled the functions a, b and c in some random order. Which of these functions correspond to which? [You do not need to show any work.]



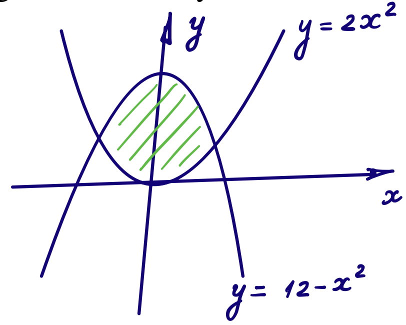
The functions (a, b, c) correspond to ...

- (A) (f, f', f'')
- (B) (f, f'', f')
- (C) (f', f, f'')
- (D) (f', f'', f)
- (E) (f'', f, f')
- (F) (f'', f', f)

Dictionary: $f' > 0 \Leftrightarrow f$ - increasing ; $f' < 0 \Leftrightarrow f$ - decreasing
$f'' > 0 \Leftrightarrow f$ - concave up ; $f'' < 0 \Leftrightarrow f$ - concave down
$f'(x) = 0 \Leftrightarrow x$ - local extremum ; $f''(x) = 0 \Leftrightarrow x$ - inflection point

_____ PTS 13. [5 pts] What is the area of the plane region bounded by the curves $y = 2x^2$ and $y = 12 - x^2$. Draw the graphs, shade the region, and show your work.

- (A) 32
- (B) 16
- (C) 8
- (D) 0
- (E) None of the above



• Points of intersection:

$$12 - x^2 = 2x^2 \Rightarrow x = \pm 2$$

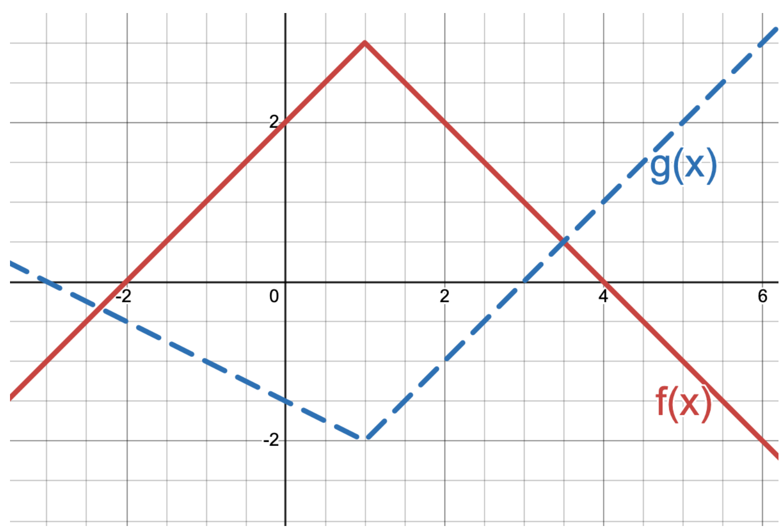
• Bounded region is symmetric w.r.t. y-axis thus:

$$\text{Area} = 2\int_0^2 (12 - x^2) - 2x^2 \, dx = 2\int_0^2 12 - 3x^2 \, dx = 2\left[12x - x^3\right]_0^2 = 2\left[24 - 8\right] = 32.$$

_____ PTS 14. [5 pts] Given the graph of f and g below, determine:

$$(f \circ g)'(2) + (fg)'(-1)$$

- (A) 0
- (B) $-1/2$
- (C) $1/2$
- (D) 1
- (E) $3/2$



$$(f \circ g)'(2) + (fg)'(-1)$$

$$= \underbrace{f'(\underbrace{g(2)}_{=-1})}_{=1} \cdot \underbrace{g'(2)}_{=1} + \underbrace{f'(-1)}_{=1} \cdot \underbrace{g(-1)}_{=-1} + \underbrace{f(-1)}_{=1}\underbrace{g'(-1)}_{=-1/2}$$

$$= 1 - 1 - \frac{1}{2}$$

$$= -1/2.$$

let $u = 25 - x^2$

$du = -2x\,dx$

$x = -4 \,..\, 3 \iff u = 9 \,..\, 16$

_____ PTS 15. [5 pts] Evaluate $\int_{-4}^{3} 6x\sqrt{25-x^2}\,dx$.

Ⓐ -182

Ⓑ -74

Ⓒ 0

Ⓓ 144

Ⓔ 288

$$= \int_9^{16} -3\sqrt{u}\,du$$

$$= -\cancel{3} \cdot \frac{2}{\cancel{3}} u^{3/2} \Big|_9^{16}$$

$$= -2\left[16^{3/2} - 9^{3/2}\right]$$

$$= -2\left[64 - 27\right]$$

$$= -2 \cdot 37$$

$$= -74.$$

_____ PTS 16. [5 pts] Suppose $h(x)$ is an even function for which $\int_0^3 h(x)\,dx = 5$ and $\int_2^3 h(x)\,dx = 1$. Find $\int_{-2}^{2} h(x)\,dx$.

Ⓐ 0

Ⓑ 3

Ⓒ 4

Ⓓ 6

Ⓔ 8

Since $h(x)$ is even,

$$\int_{-2}^{2} h(x)\,dx = 2\int_0^2 h(x)\,dx$$

$$= 2\left[\underbrace{\int_0^3 h(x)\,dx}_{=5} + \underbrace{\int_3^2 h(x)\,dx}_{=-1}\right]$$

$$= 8.$$

Part II: Free Response

Present all work leading to your final answer clearly and in a structured manner. Show all relevant steps and justify them.

17. [10 pts] Compute the following limits:

______ 3 PTS (a) $\lim_{x\to 1} \sin\left(\frac{5\pi x}{4}\right)$

$= \sin\left(\frac{5\pi}{4}\right)$

$= -\frac{\sqrt{2}}{2}$.

______ 3 PTS (b) $\lim_{x\to 0} \frac{\sin(x)}{\arcsin(2x)}$ form : $\frac{0}{0}$

[by L'Hôpital] $\stackrel{\text{L}}{=} \lim_{x\to 0} \dfrac{\cos x}{\frac{2}{\sqrt{1-4x^2}}}$

$= \frac{1}{2}$.

______ 4 PTS (c) $\lim_{x\to 1^+} x^{1/(x-1)}$ form : 1^∞

[by continuity] $= e^{\underbrace{\lim_{x\to 1^+} \frac{\ln x}{x-1}}_{\text{form : } \frac{0}{0}}}$

[by L'Hôpital] $= e^{\lim_{x\to 1^+} 1/x}$

$= e$.

______ PTS 18. [5 pts] Consider the function $f(x) = \cos(x)$.

______ 3 PTS (a) Find its linearization at $x = \pi$.

$$f'(x) = -\sin(x)$$

$$L_\pi(x) = f(\pi) + f'(\pi)(x-\pi)$$
$$= \cos(\pi) - \sin(\pi)(x-\pi)$$
$$= -1.$$

______ 2 PTS (b) Use the above linearization to estimate $\cos(3)$.

$$\cos(3) \approx L_\pi(3) = -1.$$

______ PTS 19. [5 pts] The limit below represents the **derivative** of which function at which point? Justify your answer.

$$\lim_{h\to 0} \frac{\ln(e+h)-1}{h}$$

Recall : $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

We can take $f(x) = \ln x$.

Thus, the above limit represents $\dfrac{d}{dx}(\ln(x))\Big|_{x=e}$.

______ PTS 20. [5 pts] Express the limit below as a **definite integral** over the interval $[0, \frac{\pi}{4}]$. Justify your answer. (Note that you do not need to compute the integral nor the limit.)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{\pi}{4n}}_{\Delta x} \underbrace{\tan\frac{i\pi}{4n}}_{f(x_i^*)}$$

$$= \int_0^{\pi/4} \tan(x)\, dx$$

______ PTS 21. [10 pts] On the axes provided, sketch the graph of a single function $h(x)$ which has all of the following properties :

(a) The domain of $h(x)$ is $[-3, 5)$.

(b) $h(-1) = -4$, $h(0) = 2$.

(c) $h(x)$ has a removable discontinuity at $x = -1$, and $h(x)$ is continuous through the rest of the domain.

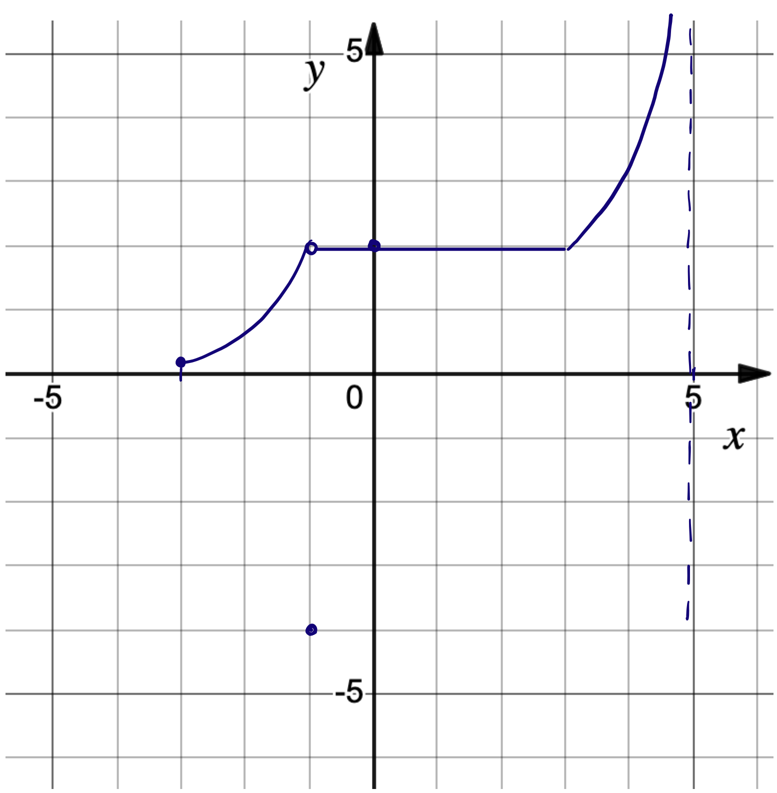
(d) $\lim_{x \to -1^-} h(x) = 2$.

(e) On the interval $(-3, -1)$, $h'(x) > 0$ and $h''(x) > 0$.

(f) On the interval $(-1, 3)$, $h'(x) = 0$.

(g) $h'(3)$ does not exist.

(h) $\lim_{x \to 5^-} h(x) = \infty$. $\Rightarrow$ $x = 5$ V.A.



22. [15 pts] Evaluate the following integrals:

_______ 5 PTS (a) $\int \left(4x^3 + \sqrt[3]{x^4} + \frac{1}{x^3}\right) dx$

$$= x^4 + \frac{3}{7} x^{7/3} - \frac{1}{2x^2} + C.$$

_______ 5 PTS (b) $\int \frac{\ln(x)}{x} dx$

let $\ln x = u$

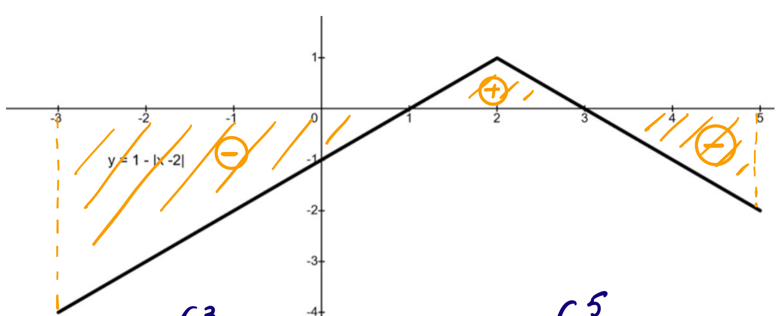
$\frac{1}{x} dx = du$

$$= \int u \, du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{\ln^2 x}{2} + C$$

_______ 5 PTS (c) $\int_{-3}^{5} 1 - |x-2| \, dx$. (Hint: The integrand is graphed below:)



$$= \underbrace{\int_{-3}^{1} 1-|x-2| \, dx}_{-A} + \underbrace{\int_{1}^{3} 1-|x-2| \, dx}_{A} + \underbrace{\int_{3}^{5} 1-|x-2| \, dx}_{-A}$$

Δ with base = 4, height = 4

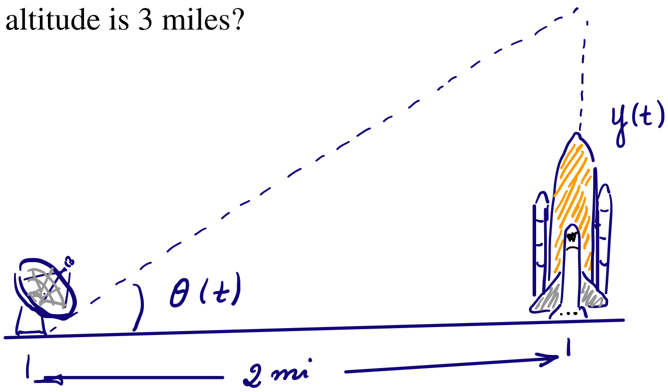
Δ with base = 2, height = 1

Δ with base = 2, height = 2

$$= \underbrace{-\frac{1}{2} 4 \cdot 4}_{=-8} + \underbrace{\frac{1}{2} 2 \cdot 1}_{=1} - \underbrace{\frac{1}{2} 2 \cdot 2}_{=-2}$$

$$= -9.$$

_____ PTS 23. [10 pts] A space shuttle is launched vertically from the JFK Center, and its position is being tracked by a radar device located 2 miles horizontally from the launch site. The shuttle is ascending vertically at a rate of 4 miles per second. Let θ represent the angle between the horizontal ground and the line of sight from the radar to the shuttle. At what rate is θ changing when the shuttle's altitude is 3 miles?



Given:

$x = 2$ mi (fixed)

$\frac{dy}{dt} = 4$ mi/s

Question:

$\frac{d\theta}{dt} = ?$ when $y = 3$ mi

Governing relation.

$y(t) = 2\tan(\theta(t))$ where $y(t)$ represents the height of the space shuttle at time t and $\theta(t)$ the angle of elevation as a function of time.

Differentiating both sides with respect to t we obtain:

$$\frac{dy}{dt} = 2\sec^2\theta \, \frac{d\theta}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{dy}{dt} \cdot \frac{\cos^2\theta}{2}$$

When $y = 3$, $\cos^2\theta = \left(\frac{2}{\sqrt{13}}\right)^2 = \frac{4}{13}$

Thus,

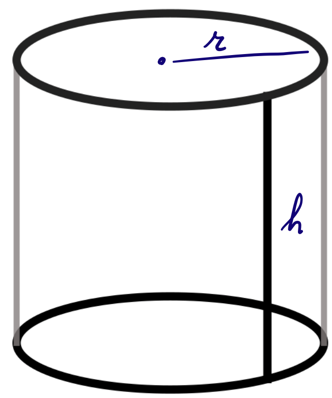
$$\frac{d\theta}{dt} = 4 \cdot \frac{1}{2} \cdot \frac{4}{13}$$

$$= \frac{8}{13}$$

The angle θ is increasing at a rate of $\frac{8}{13}$ /sec.

_____ PTS 24. [10 pts] A company is designing a new underground water tank in the shape of a right circular cylinder. The tank will be constructed using two circular steel plates for the top and bottom and a curved steel sheet for the sides. The steel sheets are joined at the seams, which must be minimized to reduce construction costs and improve durability. The tank must hold exactly $4m^3$ of water. Find the dimensions (radius and height) of the tank that minimize the total length of the seams. Note that the seams are indicated in bold.

_____ 2 PTS (a) Label the diagram of the tank, clearly marking all variables used to describe the function that you are optimizing.



_____ 3 PTS (b) Write all the constraints and equations relevant to the situation.

Constraint:

$$V = 4 = \pi r^2 h \Rightarrow h = \frac{4}{\pi r^2}$$

Equation to be optimized:

$$L = 2 \cdot 2\pi r + h$$

$$= 4\pi r + h$$

______ 2 PTS (c) Write a formula of the function you want to optimize in terms of a single variable, and give a reasonable domain for the function (this is the domain where the problem makes sense).

$$L = 4\pi r + \frac{4}{\pi r^2}$$

$$r \in (0, \infty)$$

______ 3 PTS (d) Use calculus to minimize the function and find the tank dimensions that result in the lowest construction cost.

$$\frac{dL}{dr} = 4\pi + \frac{4}{\pi}\left(-\frac{2}{r^3}\right)$$

$$= \frac{4\pi^2 r^3 - 8}{\pi r^3}$$

$= 0$: $4\pi^2 r^3 = 8 \Rightarrow r = \sqrt[3]{\frac{2}{\pi^2}}$

DNE: $r = 0 \notin dom(r)$

Check that $r = \sqrt[3]{\frac{2}{\pi^2}}$ results in the lowest construction cost:

r:		$\sqrt[3]{\frac{2}{\pi^2}}$	
sign of L'	$-$	0	$+$
		min.	

Thus, to minimize construction cost, $r = \sqrt[3]{\frac{2}{\pi^2}}$ m and

$h = \dfrac{4}{\pi\left(\frac{2}{\pi^2}\right)^{2/3}}$ m.

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