By providing my signature below I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else:

Name (sign): Solutions Name (print): ____________

Student Number: ____________

Instructor's Name: ____________ Class Time: ____________

Problem Number	Points Possible	Points Made
1	25	
2	25	
3	10	
4	15	
5	10	
6	10	
7	10	
8	10	
9	15	
10	20	
11	15	
12	10	
13	10	
14	10	
15	2	
Total:	197	

- If you need extra space use the last page.
- Please show your work. **An unjustified answer may receive little or no credit.**
- Your work must be **neat**. If I can't read it (or can't find it), I can't grade it.
- The total number of possible points that is assigned for each problem is shown here. The number of points for each subproblem is shown within the exam.
- Please turn off your mobile phone.
- You are only allowed to use a TI-30 calculator. No other calculators are permitted.
- A calculator is not necessary, but numerical answers should be given in a form that can be directly entered into a calculator.

1. Determine the first derivative of each of the following functions. Print your answer in the box provided.

____ (a) [5 pts] $f(x) = x\cos(5x) + x^2 + 8.$

$f'(x) =$ $-5x\sin(5x) + \cos(5x) + 2x$

____ (b) [5 pts] $g(t) = e^{3t^2} \cdot \ln(8t+1) + 2001.$

$g'(t) =$ $e^{3t^2} \cdot \frac{8}{8t+1} + \ln(8t+1)e^{3t^2} \cdot 6t$

____ (c) [5 pts] $F(y) = \dfrac{y}{y^2+1}.$ $= y(y^2+1)^{-1}$

$F'(y) =$ $\frac{(y^2+1)(1) - y(2y)}{(y^2+1)^2} = y(-(y^2+1)^{-2} \cdot 2y) + (y^2+1)^{-1} \cdot 1$

(d) [5 pts] $H(u) = \arccos\left(\frac{u}{2}\right)$.

$H'(u) = \frac{-1}{\sqrt{1-\left(\frac{u}{2}\right)^2}} \cdot \frac{1}{2} = \frac{-1}{\sqrt{\frac{1}{4}(4-u^2)}} \cdot \frac{1}{2} = \frac{-1}{\sqrt{4-u^2}}$

(e) [5 pts] $h(x) = (5x^2+1)^{4x}$.

$\ln(h(x)) = 4x\ln(5x^2+1)$

$\frac{1}{h(x)} \cdot h'(x) = 4x \cdot \frac{10x}{5x^2+1} + 4\ln(5x^2+1)$

$h'(x) = h(x)\left[\frac{40x^2}{5x^2+1} + 4\ln(5x^2+1)\right]$

$h'(x) = (5x^2+1)^{4x}\left(\frac{40x^2}{5x^2+1} + 4\ln(5x^2+1)\right)$

alternate method:

$h(x) = e^{4x\ln(5x^2+1)}$

$h'(x) = e^{4x\ln(5x^2+1)} \cdot \left(4x \cdot \frac{10x}{5x^2+1} + 4\ln(5x^2+1)\right)$

$h'(x) = (5x^2+1)^{4x}\left(\frac{40x^2}{5x^2+1} + 4\ln(5x^2+1)\right)$

$h'(x) = (5x^2+1)^{4x}\left(\frac{40x^2}{5x^2+1} + 4\ln(5x^2+1)\right)$

2. Determine **the most general anti-derivative** for each of the following expressions. Print your answer in the box provided.

______ (a) [5 pts] $\int (10x^4 - e^x + 2001)\ dx.$

Anti-Derivative: $2x^5 - e^x + 2001x + C$

______ (b) [5 pts] $\int \left(\cos(8x) - \frac{1}{x}\right)\ dx.$

Anti-Derivative: $\frac{1}{8}\sin(8x) - \ln|x| + C$

______ (c) [5 pts] $\int x \sin(4x^2 - 1)\ dx.$

Anti-Derivative: $-\frac{1}{8}\cos(4x^2-1) + C$

(d) [5 pts] $\int \frac{x}{\sqrt{1+x^2}}\,dx.$ $= \frac{1}{2}\int \frac{1}{\sqrt{1+x^2}} \cdot 2x\,dx = \frac{1}{2}\int u^{-1/2}du = \sqrt{u} + C$

$u = 1+x^2$
$du = 2x\,dx$

Anti-Derivative: $\sqrt{1+x^2} + C$

(e) [5 pts] $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$ $= \int e^{\sqrt{x}} \cdot x^{-1/2}dx = 2\int e^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2}dx$

$u = \sqrt{x}$
$du = \frac{1}{2}x^{-1/2}dx$

$= 2\int e^{u}du = 2e^{u} + C$

Anti-Derivative: $2e^{\sqrt{x}} + C$

3. Determine the value of each of the following limits. Show all of your work and justify your conclusions. Indicate if a limit approaches ∞ or $-\infty$ otherwise print DNE if the limit does not exist.

_______ (a) [5 pts]

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} \overset{LH}{=} \lim_{x\to 1} \frac{1/x}{1} = \boxed{1}$$

IF $\frac{0}{0}$

_______ (b) [5 pts]

$$\lim_{x\to 1^+} \frac{x^2-5x+5}{(x-1)(x+2)} = \boxed{\infty}$$

As $x \to 1^+$, the numerator approaches 1, while the denominator approaches 0 and is positive. That means the overall limit is ∞.

4. The following questions refer to the limit definition of the derivative.

(a) [5 pts] State the limit definition of the derivative of the function $f(x)$ at the point x_0.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

(b) [10 pts] Use the **limit definition of the derivative** to show that the derivative of $f(x) = 3x^2 - 4x + 10$ at x_0 is $f'(x_0) = 6x_0 - 4$.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{3(x_0+h)^2 - 4(x_0+h) + 10 - 3(x_0)^2 + 4x_0 - 10}{h}$$

$$= \lim_{h \to 0} \frac{3(x_0)^2 + 6x_0h + 3h^2 - 4x_0 - 4h + 10 - 3(x_0)^2 + 4x_0 - 10}{h}$$

$$= \lim_{h \to 0} \frac{6x_0h + 3h^2 - 4h}{h}$$

$$= \lim_{h \to 0} \frac{h(6x_0 + 3h - 4)}{h}$$

$$= \lim_{h \to 0} (6x_0 + 3h - 4)$$

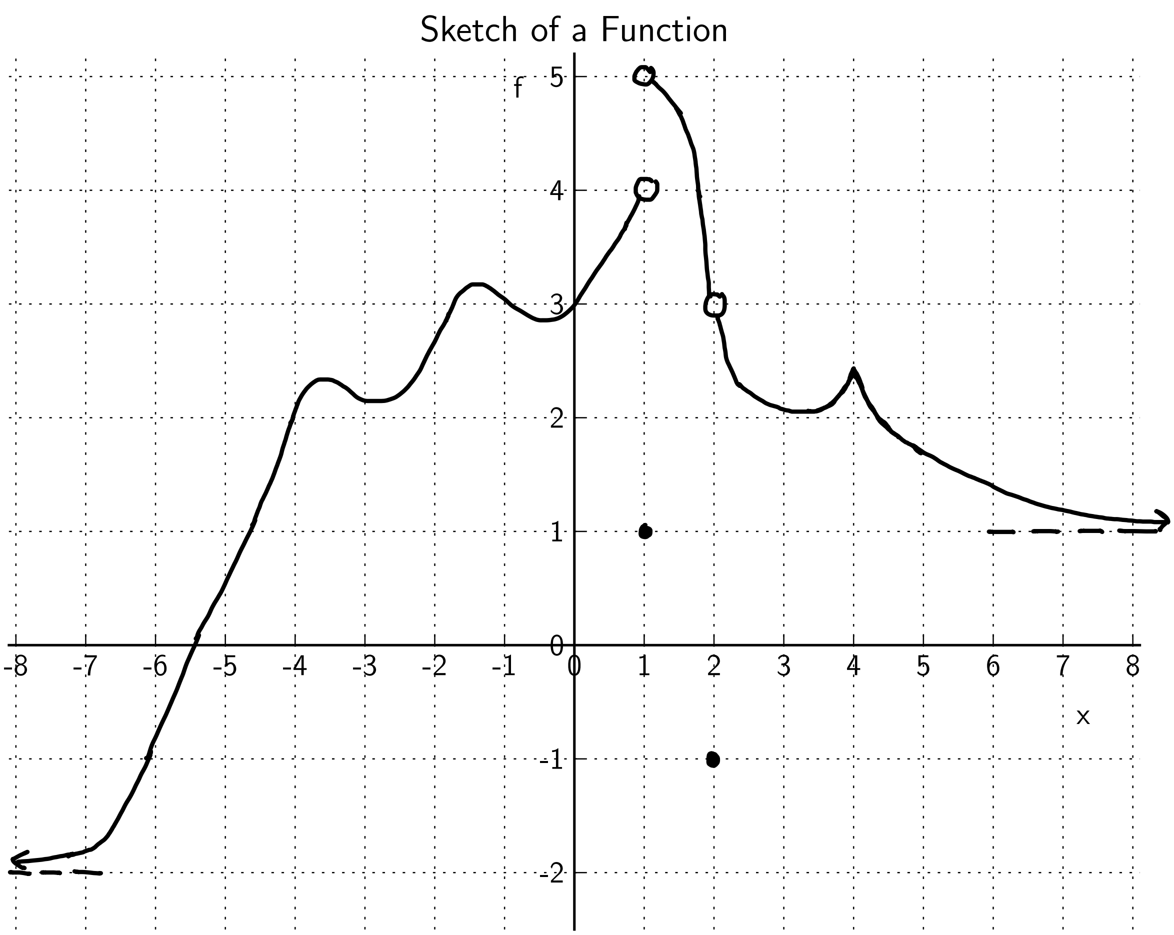
$$= 6x_0 - 4$$

Points earned: ______
out of a possible 15 points

5. [10 pts] Use the axes below to make a sketch of the graph of a function whose domain is $(-\infty, \infty)$ that satisfies all of the following criteria:

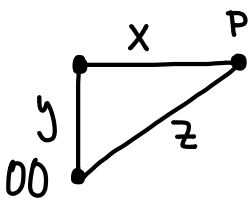
$$\begin{aligned} \lim_{x\to-\infty} f(x) &= -2, \\ \lim_{x\to1^-} f(x) &= 4, \\ f(1) &= 1, \\ \lim_{x\to1^+} f(x) &= 5, \\ \lim_{x\to2} f(x) &= 3, \\ f(2) &= -1, \\ \lim_{x\to\infty} f(x) &= 1, \end{aligned}$$

$f(x)$ is **not** differentiable at $x = 4$.



Points earned: ______
out of a possible 10 points

6. [10 pts] Popeye and Olive Oyl will be sailing away from the same dock on the small island they call home. Popeye will sail directly East at 5 knots starting at 1pm. (One knot is one nautical mile per hour.) One hour later (2pm) Olive Oyl will leave the island sailing directly South at 4 knots. Determine the rate of change of the distance between them at 3pm.



goal: $\frac{dz}{dt}$

$x = (5 \text{ knots})(2 \text{ hrs}) = 10$ nautical miles

$y = (4 \text{ knots})(1 \text{ hour}) = 4$ nautical miles

$z = \sqrt{10^2 + 4^2} = \sqrt{116}$ nautical miles

$\frac{dx}{dt} = 5$ knots

$\frac{dy}{dt} = 4$ knots

Equation: $x^2 + y^2 = z^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} = \boxed{\frac{10\cdot 5 + 4\cdot 4}{\sqrt{116}} \text{ knots}}$$

$$= \frac{33}{\sqrt{29}} \text{ knots}$$

7. [10 pts] Determine the absolute minimum and absolute maximum of the function

$$f(x) \quad = \quad x^2 e^{-x} - 3e^{-x},$$

over the interval $-2 \leq x \leq 2$.

$$f'(x) = x^2 \cdot -e^{-x} + e^{-x} \cdot 2x + 3e^{-x} = -e^{-x}(x^2 - 2x - 3) = -e^{-x}(x-3)(x+1)$$

critical numbers: $x = \cancel{3}, -1$

not in $[-2,2]$

x	$f(x)$
-2	$4e^2 - 3e^2 = e^2 \approx 7.389$
-1	$e^1 - 3e^1 = -2e^1 \approx -5.437$
2	$4e^{-2} - 3e^{-2} = e^{-2} \approx 0.135$

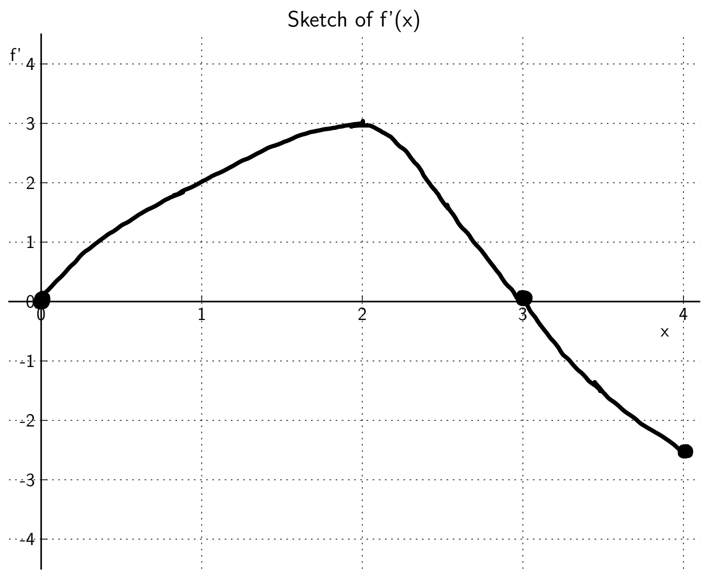
abs max: e^2

abs min: $-2e$

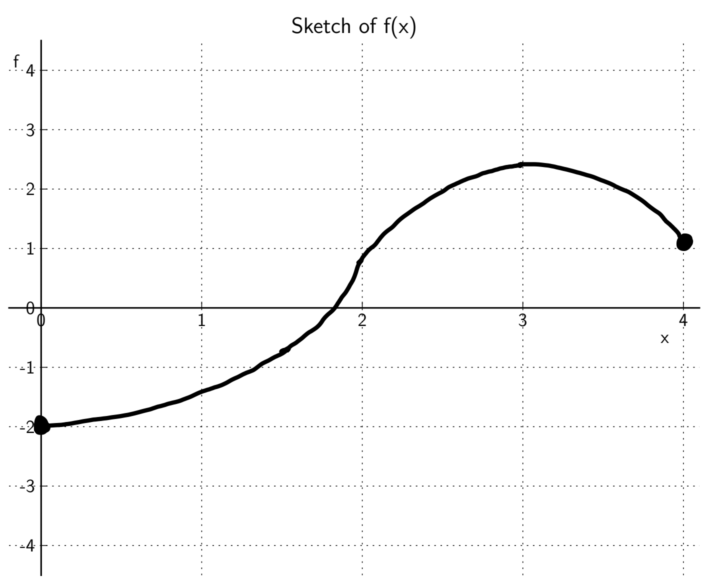
8. A function, $f(x)$, and its derivative, $f'(x)$, are both defined and continuous on the interval $0 \leq x \leq 4$. They satisfy the following criteria:

- $f'(0) = 0$, and $f'(3) = 0$.
- $f'(x)$ is strictly increasing on the interval $0 \leq x < 2$.
- $f'(x)$ is strictly decreasing on the interval $2 < x \leq 4$.
- $f(0) = -2$.

_______ (a) [4 pts] Provide a sketch of the graph $f'(x)$ that is consistent with the conditions above.

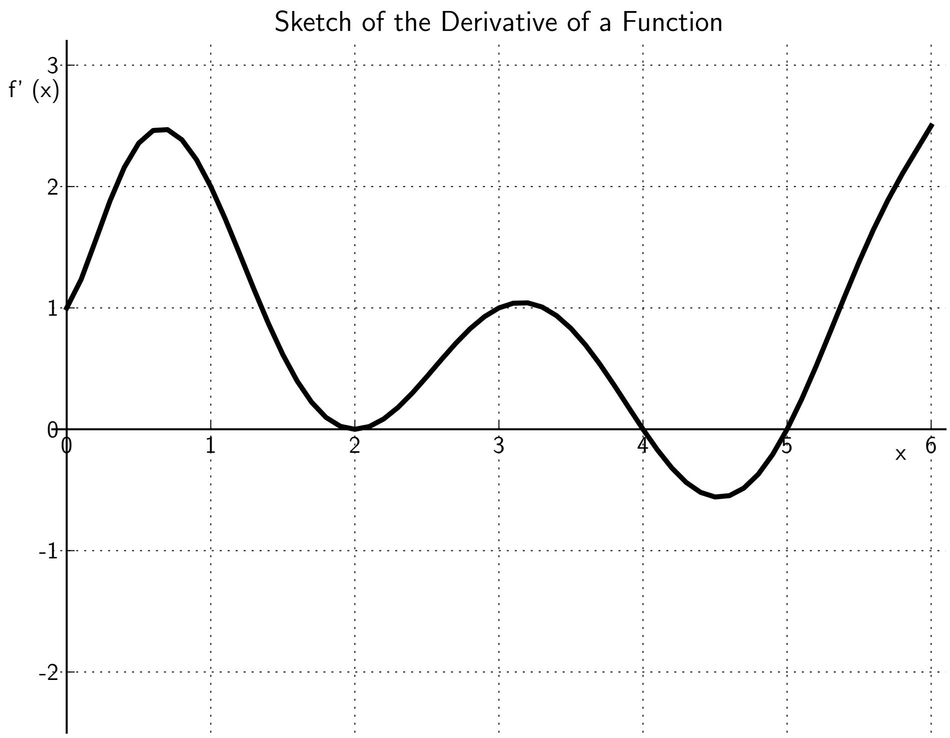


_______ (b) [6 pts] Provide a sketch of the graph of $f(x)$ that is consistent with your sketch above.



 Points earned: _____
out of a possible 10 points

9. The graph of the derivative, $f'(x)$, of a function is given below. Use the graph to answer each of the questions below. For each question provide a brief, one sentence justification for your answer.



_______ (a) [5 pts] Values of x where $f(x)$ is increasing.

$(0,4) \cup (5,6)$ f is increasing where $f' > 0$ (and also at $x=2$ since $f' > 0$ on both sides of 2 and f is continuous). (also correct: $[0,4] \cup [5,6]$)

(b) [5 pts] Values of x where $f(x)$ is decreasing.

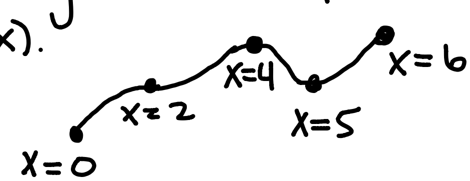
_______ $f'(x) < 0$ on $(4,5)$ so f is decreasing on $(4,5)$ (also correct: $[4,5]$)

(c) [5 pts] Value(s) of x where $f(x)$ has a local maxima or minima.

Max: 4, 6

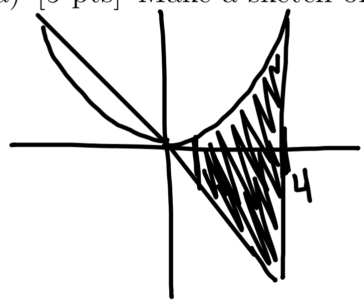
_______ Min: 0, 5

See the rough sketch of the graph of $y = f(x)$.



10. Consider the region that is bounded on the left by $x = 1$, bounded on the right by $x = 4$, bounded above by $f(x) = x^2$, and bounded below by $g(x) = -x$.

_______ (a) [5 pts] Make a sketch of the region.



_______ (b) [5 pts] Construct an estimate of the area of the region using a Riemann sum with four rectangles of equal length. Do not provide the number but provide an expression that can be directly plugged into a calculator. Also, state if the sum is a left sided sum, right sided sum, or mid-point sum for your estimate.

Exact area is $\int_1^4 x^2-(-x)\,dx = \int_1^4 (x^2+x)dx$ so use a Riemann sum for $h(x)=x^2+x$. Using right endpoints and 4 subintervals:

$\Delta x = \frac{3}{4}$ $[1, 7/4]$ $[7/4, 10/4]$ $[10/4, 13/4]$ $[13/4, 4]$

$$\boxed{\left[\left(\tfrac{7}{4}\right)^2+\tfrac{7}{4}\right]\cdot\tfrac{3}{4}+\left[\left(\tfrac{10}{4}\right)^2+\tfrac{10}{4}\right]\cdot\tfrac{3}{4}+\left[\left(\tfrac{13}{4}\right)^2+\tfrac{13}{4}\right]\cdot\tfrac{3}{4}+\left[4^2+4\right]\cdot\tfrac{3}{4}}$$

_______ (c) [10 pts] Generalize the estimate for the area to a Riemann sum with N rectangles of equal width.

$\Delta x = \frac{4-1}{N} = \frac{3}{N}$; kth right endpoint is $c_k = 1+\frac{3k}{N}$

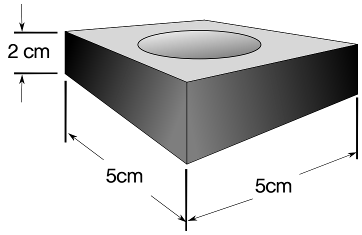
The area is approximately $\sum_{k=1}^{N}\left(\left(1+\frac{3k}{N}\right)^2+\left(1+\frac{3k}{N}\right)\right)\cdot\frac{3}{N}$

$$= \frac{3}{N}\left(\sum_{k=1}^{N} 2+\frac{9k}{N}+\frac{9k^2}{N^2}\right) = \frac{3}{N}\left(\sum_{k=1}^{N} 2 + \frac{9}{N}\sum_{k=1}^{N} k + \frac{9}{N^2}\sum_{k=1}^{N} k^2\right)$$

$$= \frac{3}{N}\left(2N + \frac{9}{N}\cdot\frac{N(N+1)}{2} + \frac{9}{N^2}\cdot\frac{N(N+1)(2N+1)}{6}\right)$$

$$= 6 + \frac{27}{2}\cdot\frac{N+1}{N} + \frac{27}{6}\,\frac{N+1}{N}\cdot\frac{2N+1}{N}$$

11. A bushing is constructed by drilling a cylindrical hole of radius r through a rectangular block. The base of the block is 5cm by 5cm, and the height of the block is 2cm.



_______ (a) [5 pts] Determine the volume, $V(r)$, of the bushing as a function of r.

$$V = lwh - \pi r^2 h$$
$$V = 50 - \pi \cdot r^2 \cdot 2$$

$V(r)$: $50 - 2\pi r^2$

_______ (b) [5 pts] Determine the linearization, $L(r)$, of the volume at $r = 2$ cm.

$$V(2) = 50 - 8\pi$$
$$V'(r) = -4\pi r$$
$$V'(2) = -8\pi$$
$$L(r) = V(2) + V'(2)(r-2)$$

$L(r)$: $50 - 8\pi - 8\pi(r-2)$

_______ (c) [5 pts] Use the linearization to estimate the error in the volume if the tolerance in the radius of the drill bit is ± 0.1cm.

$$\Delta V \approx L(2 \pm 0.1) - L(2)$$
$$= (50 - 8\pi - 8\pi(\pm .1)) - (50 - 8\pi)$$
$$= -8\pi(\pm .1)$$
$$= \pm .8\pi$$

OR

$$\Delta V \approx dV$$
$$dV = -4\pi r \, dr$$
$$dV = -4\pi(2)(\pm 0.1)$$
$$= \pm 0.8\pi$$

Error: $\pm 0.8\pi \text{ cm}^3$

12. [10 pts] The velocity of a particle is

$$v(t) \;=\; \sin(3\pi t) + e^{-t} + 1,$$

where the unit for t is seconds, and the units for the velocity are meters per second. Determine the displacement from $t = 0$ to $t = 60$.

$$s(60) - s(0) = \int_0^{60} \left(\sin(3\pi t) + e^{-t} + 1\right) dt$$

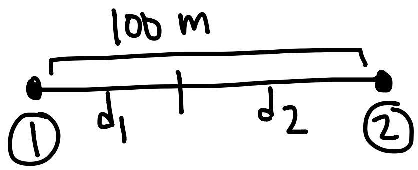
$$= \left[\frac{-1}{3\pi} \cos(3\pi t) - e^{-t} + t \right]_{t=0}^{t=60}$$

$$= \left(\frac{-1}{3\pi} \cos(180\pi) - e^{-60} + 60 \right) - \left(\frac{-1}{3\pi} \cos(0) - e^{0} + 0 \right)$$

$$= \frac{-1}{3\pi} - e^{-60} + 60 + \frac{1}{3\pi} + 1$$

$$= 61 - e^{-60} \text{ meters}$$

13. [10 pts] Two small vehicles are placed on opposite sides of a straight track facing each other. The track is 100 meters long. An experiment is conducted in which the cars move toward each other with constant velocities until they crash into one another. The cost of fuel for the first car is four times the square of the distance it moves. The cost of fuel for the second car is two times the square of the distance it moves. The experimenters want to minimize the fuel costs. How far should the first car travel? Justify that your answer is a minimum.



$C_1 = 4(d_1)^2$

$C_2 = 2(d_2)^2$

Total cost: $C = 4(d_1)^2 + 2(d_2)^2$

$d_1 + d_2 = 100$ so $d_2 = 100 - d_1$

$C = 4d_1^2 + 2(100-d_1)^2$ but d_1 isn't ideal so use x

Cost function: $C = 4x^2 + 2(100-x)^2$

domain: $[0, 100]$

$C' = 8x + 4(100-x)(-1) = 8x + 4x - 400$

$C' = 12x - 400$

Critical number: $x = \frac{400}{12} = \frac{100}{3}$

x	$C(x)$
0	$2(100)^2$
$100/3$	$4(100/3)^2 + 2\left(\frac{2\cdot 100}{3}\right)^2 = \frac{4}{9}(100)^2 + \frac{8}{9}(100)^2 = \frac{12}{9}(100)^2$ ← absolute minimum cost
100	$4(100)^2$

The first car should travel $\frac{100}{3}$ meters.

14. Consider the ellipse given by

$$x^2 + 2xy + 3y^2 = 2.$$

(a) [5 pts] Determine the equation of the tangent line to the curve at the point $(\sqrt{2}, 0)$.

$$2x + 2x\frac{dy}{dx} + y \cdot 2 + 6y\frac{dy}{dx} = 0$$

$$2x\frac{dy}{dx} + 6y\frac{dy}{dx} = -2x - 2y$$

$$\frac{dy}{dx} = \frac{-2x-2y}{2x+6y}$$

$$\left.\frac{dy}{dx}\right|_{(\sqrt{2},0)} = \frac{-2\sqrt{2}}{2\sqrt{2}} = -1$$

$$\boxed{y - 0 = -1(x - \sqrt{2})}$$

(b) [5 pts] Determine all points on the ellipse which have a horizontal tangent line.

$$-2x - 2y = 0$$

$-x = y$ and $x^2 + 2xy + 3y^2 = 2$

$$x^2 + 2x(-x) + 3(-x)^2 = 2$$

$$x^2 - 2x^2 + 3x^2 = 2$$

$$2x^2 = 2$$

$x = \pm 1$ and $y = -x$

$$\boxed{(-1, 1) \text{ and } (1, -1)}$$

15. [2 pts] How well do you think your score will be on this test compared to the other students? Circle the percentage below that corresponds to your rank. A 100% means that you will have the best score, and a 50% means that your score will be in the middle of all of the scores.

0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100

Extra space for work. **Do not detach this page.** If you want us to consider the work on this page you should print your name, instructor and class meeting time below.

Name (print): ____________ Instructor (print): ____________ Time: ____________