99 PROBLEMS

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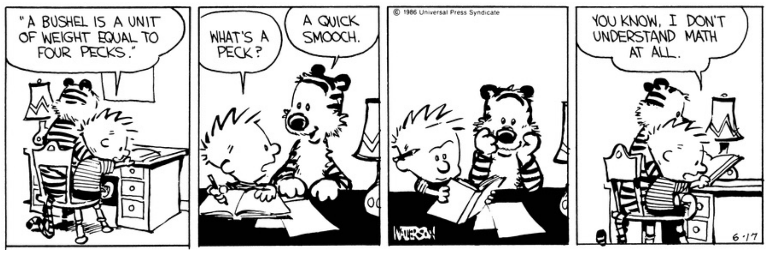

Legend: (*) harder; (**) even harder

1. Some algebra

(1) Assuming $h \neq 0$, what is $\frac{f(x+h)-f(x)}{h}$ where $f(x) = (x+1)^2$? Simplify.

(2) Find the domain of the function

$$f(x) = \frac{\sqrt{x+2} + \log_2(5-x)}{x}.$$

(3) (*) Consider the function $f(x) = \ln\left(x + \sqrt{1+x^2}\right)$. Find the domain of f. Determine the parity of this function, i.e. is it odd, even, or neither?

(4) What is the equation of the secant line joining the points of the graph $f(x) = 2^x$ whose x-coordinates are respectively 1 and 2?

(5) Find the point(s) of intersection of the hyperbolas $x^2+3xy = 54$ and $xy+4y^2 = 115$.

2. Limits

Finding the limit at a real value without using l'Hôpital's rule

(6) $\lim_{x\to 3} x^2 - 7x + 12 + \sqrt{x^2-5} =$

(7) $\lim_{x\to 2} \frac{x^2-4x+4}{x^2-5x+6} =$

(8) $\lim_{x\to 4} \frac{3-\sqrt{x+5}}{x-4} =$

(9) $\lim_{x\to 0} \frac{x^4+5x-3}{2-\sqrt{x^2+4}} =$

(10) $\lim_{x\to 1} \frac{x^3-1}{(x-1)^2} =$

(11) (*) $\lim_{x\to 0} x^4 \cos(2/x)$

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Limits of trigonometric type

(12) $\lim_{x\to 0} \frac{\sin^2 5x}{2x \tan 3x} =$

(13) $\lim_{x\to 0} \frac{\cos 2x - 1}{\cos x - 1} =$

Limits at infinity

(14) $\lim_{x\to -\infty} \frac{7}{x^3 - 4} =$

(15) $\lim_{x\to \infty} \frac{10}{x^2 + 10} =$

(16) $\lim_{x\to \infty} \frac{7x^2 + x - 100}{2x^2 - 5x} =$

(17) $\lim_{x\to \infty} x - \sqrt{x^2 + 7} =$

(18) $\lim_{x\to -\infty} \frac{7x^2 - x + 11}{4 - x} =$

(19) $\lim_{x\to \infty} \frac{x + 3}{\sqrt{9x^2 - 5x}} =$

(20) (*) $\lim_{x\to \infty} \left(\frac{x - 2}{x - 1}\right)^x =$

One sided limits

(21) $\lim_{x\to 3^+} \frac{x^2 + 3x}{9 - x^2} =$

(22) $\lim_{x\to 3^-} \frac{x^2 - 3x}{x^2 - 9} =$

3. Asymptotes

(23) The line $y = mx + p$, with $m \neq 0$ is an oblique asymptote (or slant asymptote) of $f(x)$ iff $\lim_{x\to\infty} \frac{f(x)}{x} = m$ and $\lim_{x\to\infty} f(x) - mx = p$. Show that $f(x) = \sqrt{x^2 - 4x}$ has an oblique asymptote at ∞ and a different one at $-\infty$.

(24) (*) Show that if $f(x)$ is a rational function then $f(x)$ has an oblique asymptote iff the degree of the numerator is exactly one more than the degree of the denominator. [hint: how can you write f(x) after performing polynomial division?]. Find the oblique asymptote(s) of $f(x) = \frac{x^2-6x+1}{x-2}$ using (a) the above definition and (b) using long division.

(25) (*) Can a rational function have two distinct oblique asymptotes?

Find all asymptotes (vertical, horizontal and/or oblique) of the following functions

(26) $e(x) = \dfrac{x^2 - 4x}{2x+1}$

(27) $f(x) = \dfrac{2x+1}{3x+2}$

(28) $h(x) = \dfrac{x^4+1}{x^2-1}$

(29) $i(x) = \dfrac{x^3}{x^2+1}$

(30) $j(x) = 2x - \sqrt{4x^2+4}$

(31) Find all the asymptotes (if any) to the function $f(x) = \dfrac{x^2-1}{x|x+1|}$

(32) (*) Consider the function $f(x) = ax - \sqrt{bx^2 - 1}$ where $b \geq 0$. For which value(s) of a and b does this function have an oblique asymptote of slope 5 at $-\infty$ and of slope 1 at $+\infty$?

4. Derivatives

(33) Using the limit definition, compute $f'(3)$ where $f(x) = x^2 + \frac{2}{x}$

Compute the derivatives of the following functions:

(34) $f(x) = 4x^5 - 5x^4$

(35) $g(x) = 3x^2(x^3+1)^7$

(36) $h(x) = \dfrac{(3x-1)^2}{x^2+2^x}$

(37) $i(x) = \dfrac{x}{\sqrt{1-\ln^2(x)}}$

(38) $j(x) = (\arctan(2x))^{10}$

(39) $k(x) = x^7(x^2-x)^8 \sin^4(x^2) e^{4x}$

(40) $l(x) = \arcsin\left(2^{\sin x}\right)$

(41) $m(x) = \log_5(3x^2 + x)$

(42) $n(x) = \dfrac{3\sin(x)+2}{4\sin(x)+3}$

(43) Determine the following limit quickly: $\lim\limits_{x\to 2} \dfrac{\sqrt{x^2+5}-3}{x-2}$.

(44) Find $f'\left(\frac{3\pi}{2}\right)$ where $f(x) = (\cos x + 1)^x$.

(45) If $f(2) = 3$, $g(2) = 4$, $g(3) = 2$, $f'(2) = 5$ and $g'(3) = 2$ find

$$\left(\frac{f(g(x)) + x}{f^2(2x-4)}\right)'$$

at $x = 3$.

(46) Find $\frac{dy}{dx}$ where y is a differentiable function satisfying $\frac{\sin y}{y^2+1} = 3x$.

5. Tangents

(47) Find the point of intersection of the lines tangent to the graph of $f(x) = x\sin(x)$ at $x = \frac{\pi}{2}$ and $x = \pi$.

(48) Find the tangent(s) to the graph of $f(x) = x^2 - 2x + 1$ passing through the point $(4, 1)$.

(49) Find the equation of the line tangent at $(1, 1)$ to the graph of the function

$$y^4 + xy = x^3 - x + 2.$$

(50) (*) (Legendre Transform) Consider a smooth convex function $f(x)$. Pick a slope m and let $f^*(m)$ be the y-intercept of the tangent to the graph of $f(x)$ whose slope is m. Find the function $f^*(m)$ where $f(x) = x^2 - 2x$.

6. Extrema & concavity

(51) The function $f(x) = a \ln x - a^3 x$ has a local minimum at $x = 4$ for $a \neq 0$. What is a?

(52) Over which interval is $f(x) = x^3 - 6x^2 + 3x$ (a) concave up? (b) decreasing?

7. Study of functions

Study the following functions. I.e. find the (1) domain, (2) asymptotes and/or discontinuities, study the (3) growth and (4) concavity; locate (5) all extrema and inflection points; (6) find the roots and (7) sketch the graph

(53) $x^3 - 3x^2$

(54) $x^4 - 2x^3$

(55) $\dfrac{3x+4}{2x+3}$

(56) $\dfrac{x^3}{x^2-4}$

(57) $\dfrac{x(x-3)^2}{(x-2)^2}$

(58) $\sqrt{1-x^2}$

(59) $\sqrt{x^2-1}$

(60) $3\left(\sqrt{x^2-1} - x\right)$

(61) $\dfrac{1}{x} - \dfrac{1}{x(1-x)}$

(62) $\dfrac{|x-2|}{x-3}$

(63) $\dfrac{|x-2|}{x} - 3$

(64) (*) $3x^{\frac{2}{3}} - 2x$

(65) $\sin(2x) - 2x$

(66) (*) Consider the function $f(x) = \frac{1}{x^2-3x+2}$. Study and sketch the function. Using the previous graph, plot (a) $\phi(x) = e^{f(x)}$ and, (b) $\psi(x) = f(|x|)$.

8. Varia

(67) Give a lower bound on the number of roots of $f(x) = \cos(\pi x)/x$ on the interval $[1, 3]$. [hint: Intermediate value theorem]

(68) Suppose that a function $f(x)$ has a maximum at $x = 3$. True or False? Justify.

- The function $f^2(x)$ has a maximum at 3.
- The function $e^{f(x)}$ has a maximum at 3.
- The function $f(x-3)$ has a maximum at 0.

(69) Without a calculator estimate $\sin^2\left(.99\frac{\pi}{4}\right)$.
(70) If $-1 \leq f'(x) \leq 3$ for all x in $[1,4]$ and $f(2) = 4$, find the maximal and minimal possible values of $f(4)$.
(71) (**) Suppose that $f : [0,1] \to [0,1]$ is a continuous function. Prove that f has a fixed point in $[0,1]$, i.e., there is at least one real number x in $[0,1]$ such that $f(x) = x$.
(72) (**) Suppose that g is a continuous function on $[0,2]$ satisfying $f(0) = f(2)$. Show that there is at least one real number x in $[1,2]$ with $f(x) = f(x-1)$.
(73) Suppose that $\sum_{i=1}^{10} a_i = 100$ compute $\sum_{i=1}^{10}(2a_i + 3 - i)$.

9. Related Rates

(74) A $10ft$ ladder is leaning against the wall. How fast is the bottom of the ladder sliding when the top part is $3ft$ above the ground and gliding at a rate of $1ft$ per second.
(75) A conical cup has a diameter of $4cm$ and a height of $8cm$. How fast is the level dropping when the height is $4cm$ and the water escapes from the bottom at a rate of $1cm^3$

10. Optimization

(76) Find the maximal area of rectangle whose sides are parallel to the coordinate axes and whose vertices lie on the curve of equation $x^2 + y^4 = 1$.
(77) We have 12 m^2 of material to make a box whose bottom is square and sides are rectangular (the box has no top). What is the maximal volume that such a box can have?

11. Integration

(78) Using 4 rectangles and the right endpoint method estimate $\int_0^{12} \frac{2}{x+2}\,dx$.
(79) Compute the area under the graph of $g(x) = x + 3x^3 - \sin(2x) + xe^{x^2} + x^2$ over the interval $[-3,3]$.

Compute the following integrals

(80) $\int_0^1 xe^{-x^2}\,dx$
(81) $\int (\sin x + \cos x)^2\,dx$
(82) $\int_0^1 \frac{x^4 - 3x^2}{x^2}\,dx$
(83) $\int_{-2}^{3} |x-1|\,dx$
(84) $\int \frac{x^3}{x^2+\pi}\,dx$
(85) $\int x5^{2x}\,dx$

(86) (*) $\displaystyle\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$

(87) (*) $\displaystyle\int \frac{1}{1+e^x}\,dx$

(88) (**) $\displaystyle\int \frac{1}{1+\sin^2(x)}\,dx$

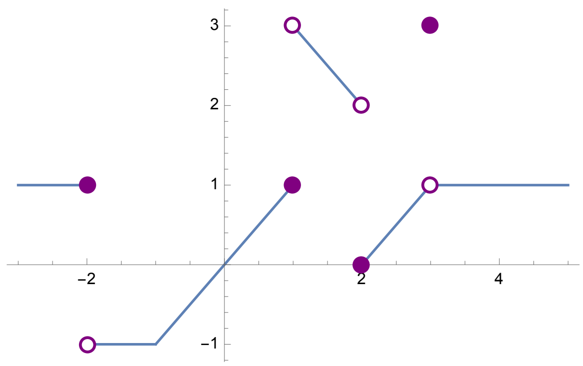
(89) Compute the area of the region between the parabolas $y = 2x^2 - 2$ and $y = x^2 + x$.

(90) Compute the area of the region bound by $y = x^3 + x$, $y = x^3$, $x = -2$ and $x = 1$.

12. Fundamental Theorem of Calculus

(91) Find $f'(x)$ where $f(x) = \displaystyle\int_x^{x^2} \frac{\sin t}{t}\,dt$.

13. Graph analysis



Based on the above picture representing the graph of $f(x)$, answer the following questions.

(92) $\displaystyle\lim_{x\to 1^+} f(x) =$

(93) $\displaystyle\lim_{x\to 3} f(x) =$

(94) (*) $\displaystyle\lim_{x\to -2^+} f(-x) =$

(95) $f'\left(\frac{3}{2}\right) =$

(96) $\displaystyle\int_{-1}^{3} f(x)\,dx$

(97) $F'(4) =$ where $F(x) = \int_0^x f(t)\,dt$

(98) Sketch $f'(x)$

(99) Sketch $F(x) = \int_1^x f(t)\,dt$

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