Division of Academic Enhancement
UNIVERSITY OF GEORGIA

Department of Mathematics
Franklin College of Arts and Sciences
UNIVERSITY OF GEORGIA

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MATH 1113 Exam 3 Review

Topics Covered

Section 4.1: Angles and Their Measure
Section 4.2: Trigonometric Functions Defined on the Unit Circle
Section 4.3: Right Triangle Geometry
Section 4.4: Trigonometric Functions of Any Angle
Section 4.5: Graphs of Sine and Cosine Functions
Section 4.7: Inverse Trigonometric Functions

Section 5.1 Fundamental Trigonometric Identities
Section 5.2 Sum and Difference Formulas

How to get the most out of this review:

1. **Watch the video and fill in the packet for the selected section. (Video links can be found at the two web addresses at the top of this page)**
2. **After each section there are some 'Practice on your own' problems. Try and complete them immediately after watching the video.**
3. **Check your answers with the key on the last page of the packet.**
4. **Go to office hours or an on-campus tutoring center to clear up any 'muddy points'.**

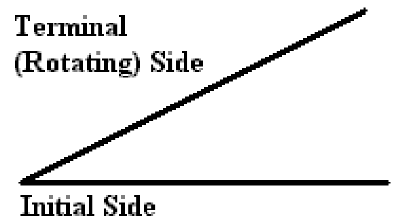
Section 4.1: Angles and Their Measure

Trigonometric Functions of Acute Angles

An acute angle is an angle with a measure satisfying:

$$0° \leq \theta < 90°$$

$$0 \leq \theta < \frac{\pi}{2}$$



An obtuse angle is an angle with a measure satisfying:

$$90° < \theta$$

$$\frac{\pi}{2} < \theta$$

A right triangle is a triangle with one 90° angle.

Angles are complementary when their sum adds up to 90°.
Angles are supplementary when their sum adds up to 180°.

Radian Measure

1 radian is the angle measure of the arc of a circle where $s = r$. $s = arclength, r = radius$

$$180° = \pi\ radians$$

To convert from radians to degrees:

$$\theta° \frac{\pi}{180°} = \theta\ (in\ radians)$$

To convert from degrees to radians:

$$\theta \frac{180°}{\pi} = \theta°$$

$\pi\ \approx 3.14$ radians

Radian Measure and Geometry

$s = r\theta$ where $s = arclength, \theta = angle$ (in radians) and $r = radius$

Area of a sector of a circle: There are three possible ways to measure depending on which two variables you know. θ MUST BE IN RADIANS!!!

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}rs = \frac{s^2}{2\theta}$$

Angular Speed	Linear Speed
$\omega = \frac{\theta}{t}$	$v = \frac{d}{t}$

$$v = r\omega$$

Example 1

Kim's donut tasting party was so successful that it propelled her YouTube hits to such a level that she was invited to compete in a bake-off competition on the Food Network. Kim won first prize and was awarded The Golden Donut. She was so excited when she won that she dropped it and it rolled 100m before she could catch it. If the diameter of The Golden Donut trophy is 0.1m, answer the following:

(a) Through what total angle did The Golden Donut rotate in radians?

(b) Through what total angle did The Golden Donut rotate in degrees?

(c) How many revolutions did The Golden Donut rotate? (Round your answer to 1 decimal place)

Example 2

A circular sector with central angle 150° has an area of 20 square units. Determine the radius of the circle.

Practice on Your Own

1. A slice of pizza comes from a 16 inch diameter pie which was cut into 7 equally sized slices. What is the area of each slice?

2. Suppose a robot has a straight arm 9 inches long. If the robot's arm sweeps (rotates) through an angle of $120°$.

   (a) Find the length of the arc swept by the arm of the robot.

   (b) Find the area of the sector swept by the arm of the robot.

Section 4.2: Trigonometric Functions Defined on the Unit Circle

The Unit Circle

The unit circle consists of all points (x, y) that satisfy the equation $x^2 + y^2 = 1$ which is a circle of radius 1. See the back page for a blank unit circle.

Example 3

In which quadrants to $\tan\theta$ and $\sin\theta$ have the same sign?

Example 4

If $\cos x = \frac{\sqrt{2}}{2}$, find $\sin x$, $\cos(-x)$ and $\tan x$.

Example 5

Assume $6\sin^2 x - 7\cos^2 x = 1$. Find the value of $\csc^2 x$.

Example 6

Give the location(s) on the unit circle where the following quantities are undefined on the interval $[0,2\pi]$.

(a) $\sec\theta$

(b) $\csc\theta$

(c) $\tan\theta$

(d) $\cot\theta$

Practice on Your Own

1. Determine the exact number that represents the values requested below. Your result should not be in terms of any trigonometric function. All angles are given in radians. Show your work and do not provide calculator approximations.
   (a) Determine the value of $\sin\alpha$ where $\cos\alpha = 0.65$, and α is in the fourth quadrant.

   (b) Determine the value of $\tan\beta$ where $\cos\beta = 0.2$ and $\pi \leq \beta \leq 2\pi$.

2. Find the value of $\cos\theta$ if the angle θ is in the third quadrant and $\sin\theta = -0.8$.

Section 4.3: Right Triangle Geometry

The following 6 Trig functions only apply to right triangles.

$\cos\theta = \frac{adj}{hyp}$

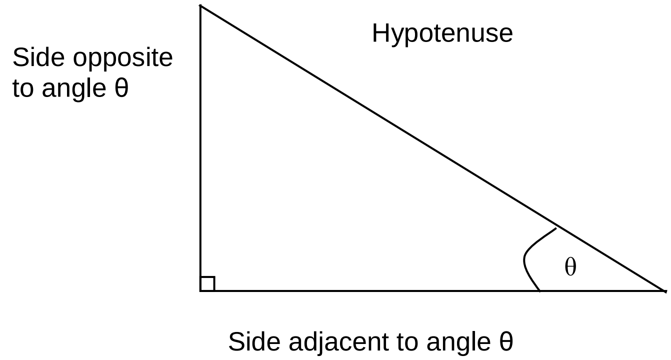
$\sin\theta = \frac{opp}{hyp}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opp}{adj}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj}$

$\csc\theta = \frac{1}{\sec\theta} = \frac{hyp}{opp}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{adj}{opp}$



Example 7

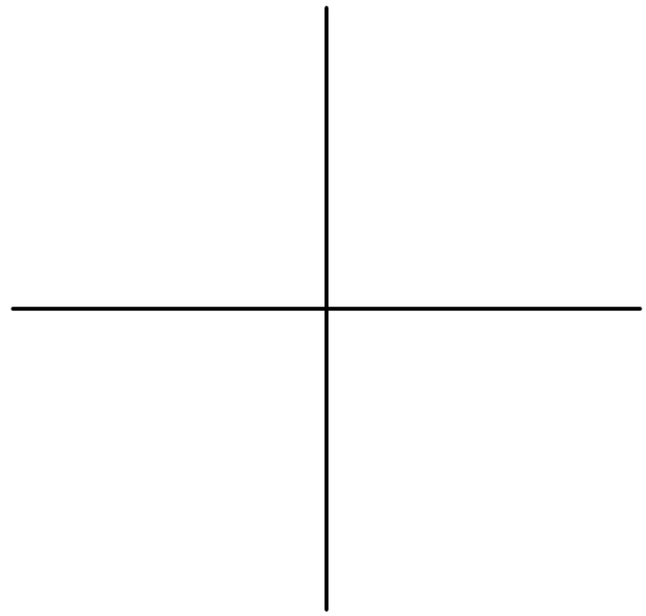
Assume θ lies in quadrant 3 and the terminal side of θ is perpendicular to the line $y = -x + 2$. Determine $\sin\theta$ and $\sec\theta$.

Example 8

Let point $A(-3,-6)$ be the endpoint of the terminal side of an angle θ in standard position. Compute the following:

(a)

$\tan\theta =$

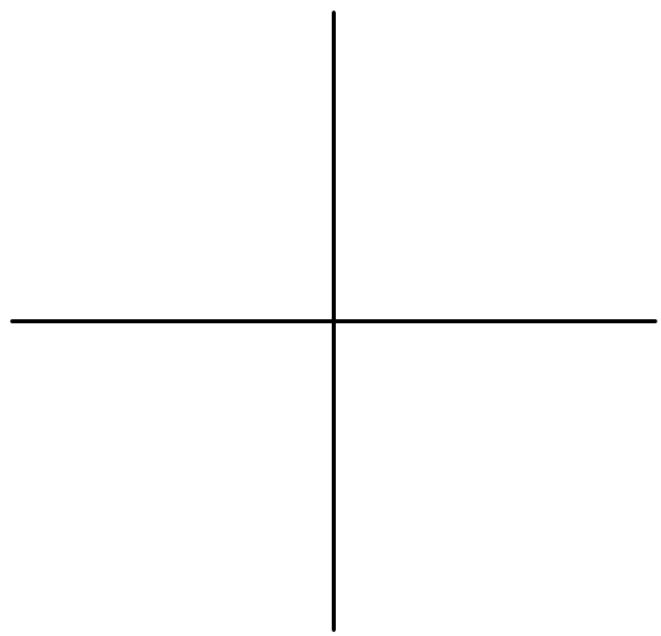


$\sec\theta =$

Let point $B(4,-7)$ be the endpoint of the terminal side of an angle α in standard position. Compute the following:

(b)

$\cot\alpha =$



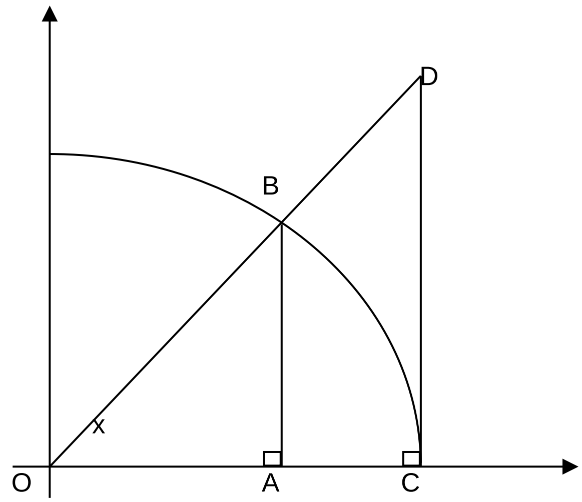
$\sin\alpha =$

Example 9

Find all possible values of $\sin x$ in the interval $[0,2\pi]$ when $\cos x = 5/7$. There may be more than one right answer.

Example 10

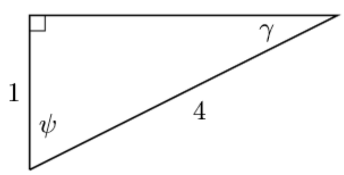
The arc in the graph is a section of the unit circle centered at (0,0). Find the lengths of CD and OD in terms of x.



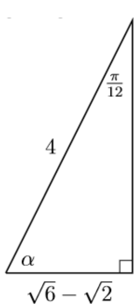
<u>Practice on Your Own</u>

1. For each diagram below determine the value of the requesting quantities. Do not give decimal approximations.

   (a) Find $\cos\varphi$, $\sin\varphi$, and $\tan\gamma$



   (b) Find α, $\sin\alpha$, and $\tan\alpha$



Section 4.4: Trigonometric Functions of Any Angle

Trigonometric Functions of Angles
Positive angles are measured from the positive x-axis rotating counterclockwise.
Negative angles are measured from the positive x-axis rotating clockwise.

Reference Angle is the acute angle ($\theta \leq \pi/2$) formed by the terminal side and the horizontal axis.

Coterminal angles are angles that share the same terminal side.

Example 11
Find the reference angle for the following:

(a)

$$\theta = \frac{11\pi}{6}$$

(b)

$$\theta = 317°$$

Example 12
Find all the exact values of x that satisfy $\tan x = 1$ on the interval $[-2\pi, 2\pi]$. Show your work or no credit will be awarded. (Hint: A diagram counts as work)

Example 13

Devin was so upset about Florida's crushing loss to Georgia this season that he climbed a tree after the game and wouldn't come down. His friends Erik and Scotti need a ladder to retrieve him but don't know how tall the tree is. If Erik is standing to the left of the tree with an angle of elevation of 21° and his distance to the top of the tree is 42m. Scotti is standing to the right of the tree with an angle of elevation of 47°. Answer the following.

(a) How long does the ladder need to be to reach Devin if their plan is to climb straight up the tree?

(b) How far apart are Erik and Scotti from each other?

Practice on Your Own

1. Determine an angle θ that matches the criterion given below. (If there are multiple answers, you only need to give one)
   (a) An angle that is coterminal with $\alpha = \pi/4$ and is greater than π

   (b) An angle that is coterminal with $\theta = 3\pi/4$ and is negative

2. A turtle sits at the edge of a circular pond of diameter 30 ft. Suppose the pond is on a coordinate plane with the center of the point at $(0,0)$ and the turtle is sitting on the positive side of the x-axis. The turtle crawls in a clockwise direction through an angle of 60 degrees. What are the coordinates of the new location of the turtle? Give exact answers.

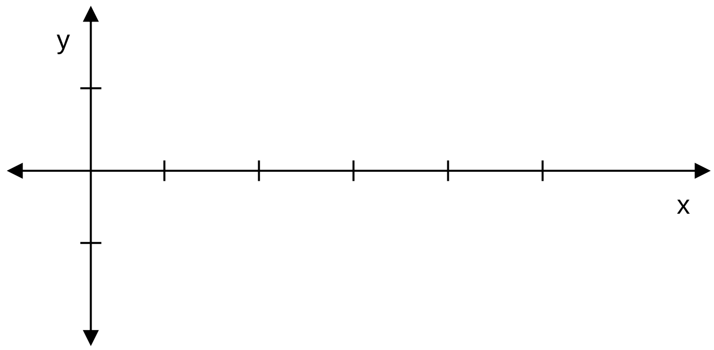
3. An elevator full of painters is moving down the edge of a skyscraper at a constant speed. You are standing one hundred feet away from the skyscraper pointing a laser at the painters. When you first start doing this, the laser beam has an angle of elevation of 33°, and ten seconds later it has an angle of elevation of 23°. What is the speed of the elevator's descent, in ft/sec?

Section 4.5: Graphs of Sine and Cosine Functions

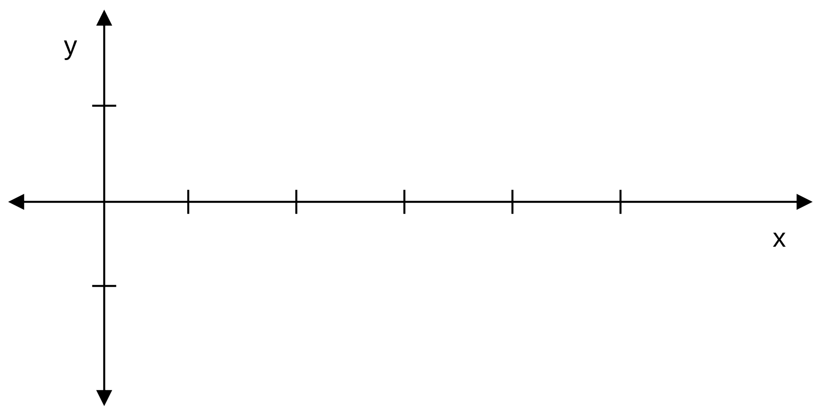
Graphs of Sine and Cosine Functions

Sketch the graphs of the 6 trigonometric functions in the space provided.

$\sin x$



$\cos x$



General Wave Properties

Waves oscillate, or bounce, between their maximum and minimum values.

$$y = A\sin[B(x - C)] + D \qquad y = A\cos[B(x - C)] + D$$

$$A = amplitude = \frac{max - min}{2} \qquad B = frequency = \frac{2\pi}{T}, T = time\ period$$

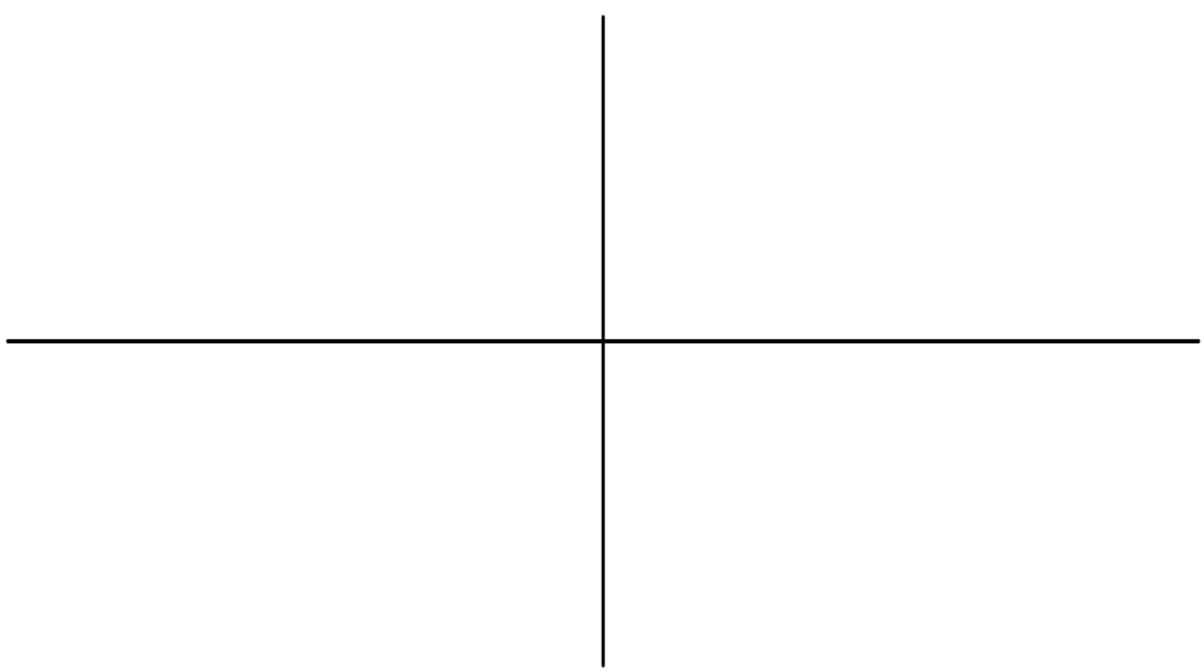
$$C = phase\ shift = horizontal\ shift \qquad D = vertical\ shift = \frac{max + min}{2}$$

Example 14

A sine wave oscillates between 6 and 10, has a period of 4π and is shifted left π.

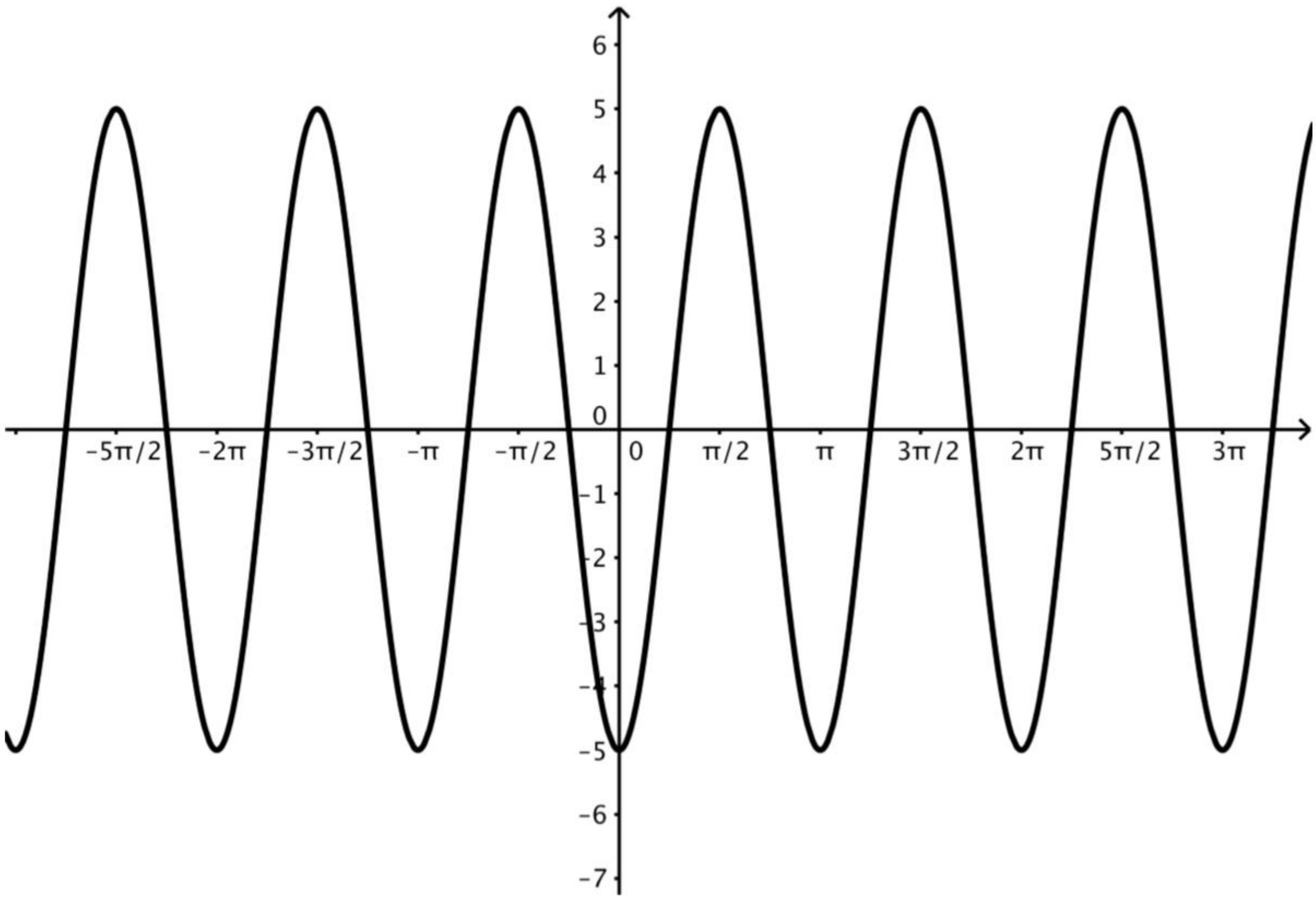
(a) Write the equation of the wave in the form $y = a\sin[bx + c] + d$.

(b) Sketch the wave you found in part (a) on the interval $[-2\pi, 2\pi]$. Clearly label the amplitude, period, y-intercept and scale the x-axis by increments of $\frac{\pi}{2}$.



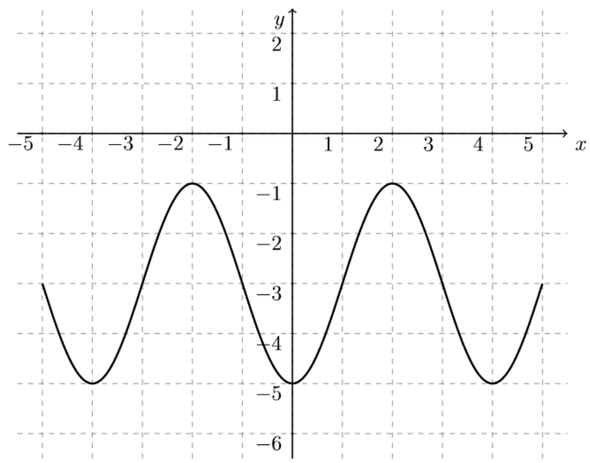
Example 15

The graph below is a graph of a function $f(x) = a\cos(2x + c)$. If $a > 0$ and $c > 0$, determine the values of a and c that would produce the graph below.



Practice on Your Own

1. The function below is defined by $f(x) = A\sin(bx - c) + d$. Determine the values of A, b, c and d where A is a positive number.



2. Refer to the graph of $y = \sin x$ to find the exact values of x in the interval $[0,4\pi]$ that satisfy the equation, $-4\sin x = -2$.

Section 4.7: Inverse Trigonometric Functions

Inverse Trig Functions and their Graphs

Sketch the graphs of the inverse trig functions in the space provided. State their respective domain and range.

$\sin^{-1}(x)$

D:

R:

$\cos^{-1}(x)$

D:

R:

$\tan^{-1}(x)$

D:

R:

Inverse cosine	arccos	the unique number in the interval $[0,\pi]$ whose cosine is x.
Inverse sine	arcsin	the unique number in the interval $[-\pi/2,\pi/2]$ whose sine is x.
Inverse tan	arctan	the unique number in the interval $[-\pi/2,\pi/2]$ whose tangent is x.

Properties of Inverse Trig Functions

$\sin(\sin^{-1} x) = x$ For every x in $[-1,1]$

$\sin^{-1}(\sin x) = x$ For every x in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\cos(\cos^{-1} x) = x$ For every x in $[-1,1]$

$\cos^{-1}(\cos x) = x$ For every x in $[0,\pi]$

$\tan(\tan^{-1} x) = x$ For all reals $(-\infty,\infty)$

$\tan^{-1}(\tan x) = x$ For every x in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Example 16
Find the exact value of the following:

(a)

$$\sin\left(\arccos\left(-\frac{2}{3}\right)\right)$$

(b)

$$\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right)$$

Example 17
Simplify the following expressions so that it only contains the variable u and contains no trigonometric or inverse trigonometric functions.

(a)

$$\tan\left(\sin^{-1}\frac{u}{\sqrt{u^2+2}}\right)$$

(b)

$$\sec\left(\operatorname{arccot}\frac{\sqrt{4-u^2}}{u}\right)$$

Practice on Your Own

1. Given the information below determine the values of the requested quantities. Please give exact values, not calculator approximations.
   (a) The point $(x, 0.3)$ is on the unit circle and in the first quadrant. Determine the value of x.

   (b) $\arctan -\sqrt{3}$

   (c) $\arcsin(\sin 5\pi/6)$

   (d) $\sin(\arccos(0.2))$

2. Simplify the following expression so that it contains only the variable u, and contains no trigonometric or inverse trigonometric functions.

$$\cos(\tan^{-1}(u) + \sec^{-1}(u))$$

Sections 5.1 & 5.2: Trig Identities

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Opposite Angle Identities

$$\cos(-\theta) = \cos\theta \qquad \sin(-\theta) = -\sin\theta \qquad \tan(-\theta) = -\tan(\theta)$$

Cosine is an even function | Sine is an odd function | Tangent is an odd function

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x} \qquad \tan x = \frac{1}{\cot x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Quotient Identities

$$\sin(u+v) = \sin u\cos v + \cos u\sin v \qquad \sin(u-v) = \sin u\cos v - \cos u\sin v$$

$$\cos(u+v) = \cos u\cos v - \sin u\sin v \qquad \cos(u-v) = \cos u\cos v + \sin u\sin v$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u\tan v} \qquad \tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u\tan v}$$

Examples 18

Verify that the following expression is an identity.

$$\frac{1}{1-\sin x} + \frac{1}{1+\sin x} = 2\sec^2 x$$

Example 19
Verify the following identity

$$\frac{\sin\theta}{\csc\theta - \cot\theta} = 1 + \cos\theta$$

Example 20
Verify the following identity

$$\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = \sqrt{2}\sin x$$

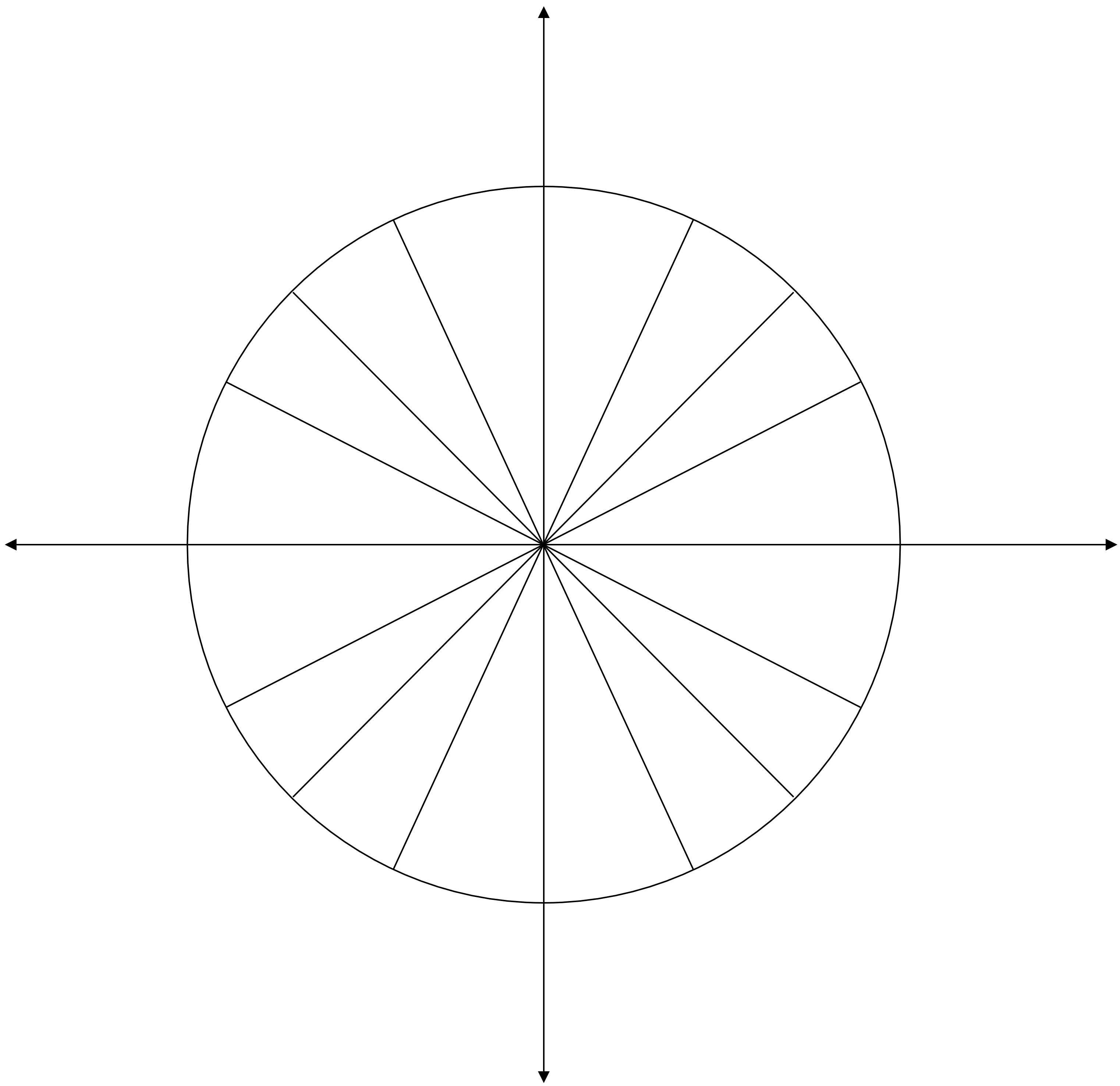
Example 21
Verify the following identity

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$$

Practice on Your Own

1. Verify the following identity: $\sin\left(\theta + \frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\sin\theta + \cos\theta)$

The Unit Circle



Answers to Practice Problems

Section 4.1

1. Pizza slice

$$A = \frac{64\pi}{7}\ in^2$$

2. Robot

   (a)

$$S = 6\pi\ in$$

   (b)

$$A = 27\pi\ in^2$$

Section 4.2

1. Trig functions

   (a)

$$\sin\alpha = -\sqrt{1 - 0.65^2}$$

   (b)

$$\tan\beta = -\frac{\sqrt{0.96}}{0.2}$$

2. Trig functions

$$\cos\theta = -\sqrt{0.36}$$

Section 4.3

1. Triangles

   (a)

$$\cos\varphi = \frac{1}{4}, \sin\varphi = \frac{\sqrt{15}}{4}, \tan\gamma = \frac{1}{\sqrt{15}}$$

   (b)

$$\alpha = \frac{5\pi}{12}, \sin\alpha = \frac{\sqrt{8 + 4\sqrt{3}}}{4}, \tan\alpha = \frac{\sqrt{8 + 4\sqrt{3}}}{\sqrt{6} - \sqrt{2}}$$

Section 4.4

1. Angles

   (a) $\theta = \frac{9\pi}{4}$ or $\frac{\pi}{4} + 2\pi n$ for any positive integer n.

   (b) $\theta = -\frac{5\pi}{4}$ or $\frac{\pi}{4} - 2\pi n$ for any positive integer n.

2. Turtle

$$\left(\frac{15}{2}, \frac{15\sqrt{3}}{2}\right)$$

3. Elevator

$$10(\tan 33° - \tan 23°)\ ft/s$$

Section 4.5

1. Wave

$$f(x) = 2\sin\left(\frac{\pi}{2}x - \frac{\pi}{2}\right) - 3$$

$$A = 2, b = \frac{\pi}{2}, c = \frac{\pi}{2}, d = -3$$

2. Sine wave

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

Note: $[0.4\pi]$ means take two counterclockwise trips around the unit circle

Section 4.7

1. Inverse Trig
   (a) $x = \sqrt{0.91}$
   (b) $x = \pi/3$
   (c) $\pi/6$
   (d) $\sqrt{0.96}$
2. Simplify

$$\frac{1}{\sqrt{u^2+1}}\left(\frac{1}{u}\right) - \frac{u}{\sqrt{u^2+1}}\left(\frac{\sqrt{u^2-1}}{u}\right) = \frac{1 - u\sqrt{u^2-1}}{u\sqrt{u^2+1}}$$

Section 5.1/5.2

1. Identity

$$\sin\left(\theta + \frac{3\pi}{4}\right) = \sin\theta\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4}\cos\theta$$

$$= -\frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta$$

$$= \frac{\sqrt{2}}{2}(-\sin\theta + \cos\theta)$$