1) Find all real numbers $x$ such that $x^3 = x$. Prove your answer!

2) a) Prove that $\sqrt{6}$ is an irrational number. You may use the fact that if an integer $x^2$ is divisible by 6, then also $x$ is divisible by 6. (For “extra credit”, prove the fact of the previous sentence using the uniqueness of prime factorizations.)
   b) Show that if $x^2$ is an irrational number, so is $x$.
   c) Show that $\sqrt{2} + \sqrt{3}$ is irrational.

3) A subset $S$ of the real numbers is dense if for any real numbers $a < b$, there exists $x \in S$ such that $a < x < b$.
   a) Show that the set of rational numbers is dense. (Suggestion: make use of the fact that every real number has a decimal expansion.)
   b) Suppose $a, b \in \mathbb{Q}$ with $b \neq 0$. Show that $a + b\sqrt{2}$ is irrational.
   c) Show that the set $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is dense.
   d) Conclude that the set of irrational numbers is dense.

4) Prove that for any real numbers $x, y$, $||x| - |y|| \leq |x - y|$.
   (Comment: this is called the Reverse Triangle Inequality, and I fear that I underplayed it in the lectures. It can be proven from the usual Triangle Inequality, for instance, or by the method of squaring both sides.)

5) State the Principle of Mathematical Induction and the Principle of Strong/Complete Induction.

6) Let $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ be $n$ different functions, and let $c \in \mathbb{R}$. Suppose that for all $1 \leq i \leq n$, $\lim_{x \to c} f_i(x) = L_i$. Show that $\lim_{x \to c} f_1(x)f_2(x)\cdots f_n(x) = L_1L_2\cdots L_n$. (Suggestion: use the product rule for limits and induction on $n$.)

7) We define a sequence of numbers $x_1, x_2, \ldots, x_n$ recursively, as follows: $x_1 = 0$, and for all $n \geq 1$, $x_{n+1} = 2x_n + 1$.
   a) Compute the first 8 terms of the sequence.
   b) Find a closed form expression for $x_n$ and prove it by induction.

8) Let $f : \mathbb{R} \to \mathbb{R}$ be a function.
   a) Show that if $f$ is odd, $f(0) = 0$.
   b) Suppose that $f$ is even and $f$ is differentiable. Give a geometric explanation of why $f'(0) = 0$. (Suggestion: say something about slopes of secant lines.)
   c) Give an example of an even function which is not differentiable at $x = 0$.

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1Perhaps this would be too difficult for an in-class exam.
9) a) Sketch the graphs of $y = |\sin x|$, $y = \sin^2 x$ and $y = \sin(\frac{1}{x}).$

b) Suppose you are given the graph of $y = f(x)$. Explain how to obtain the graph of $y = |f(x)|$.

c) Use part b) to give a geometric explanation (no $\epsilon$’s and $\delta$’s required!) of why $y = f(x)$ continuous implies $y = |f(x)|$ continuous.

d) If $y = f(x)$ is differentiable, must $y = |f(x)|$ be differentiable? If not, explain how to find the points of non-differentiability of $y = |f(x)|$. (Again, formal proofs are not required here.)

10) a) Give the $\epsilon$-$\delta$ definition of “$\lim_{x \to c} f(x) = L$”.

b) Give the $\epsilon$-$\delta$ definition of “$f$ is continuous at $x = c$”.

c) Using the definitions of parts a) and b), prove that $f$ is continuous at $c$ if and only if $\lim_{x \to c} f(x) = f(c)$.

11) Show that the following limits exist directly from the $\epsilon$-$\delta$ definition.

a) $\lim_{x \to 4} 3x - 19 = -7$.

b) $\lim_{x \to 2} x^3 = 8$. 