Applications of filtration-theoretic invariants in Floer homology to symplectic topology

Michael Usher

University of Georgia

March 14, 2009/ Third Illinois-Indiana Symplectic Geometry Conference
Outline

1. Hamiltonian Floer homology
2. The filtration
3. The Oh-Schwarz spectral invariant
4. The boundary depth
Throughout this talk, \((M, \omega)\) is a closed symplectic manifold and \(S^1 = \mathbb{R}/\mathbb{Z}\).

\[ H : S^1 \times M \rightarrow \mathbb{R} \text{ smooth} \quad \mapsto \quad \text{time-dependent vector field } X_H, \]

\[ \text{given by } \iota_{X_H} \omega = d(H(t, \cdot)) \]

\[ \mapsto \quad \text{Hamiltonian flow } \{ \phi^t_H \}_{t \in \mathbb{R}} \]

\[ \phi^0_H = \text{Id}, \quad \frac{d}{dt}(\phi^t_H(p)) = X_H(t, \phi^t_H(p)). \]

\(H\) is called non-degenerate if, for each fixed point \(p \in M\) of \(\phi^1_H\), the linearization \((\phi^1_H)_* : T_p M \rightarrow T_p M\) has all eigenvalues different from 1.
Throughout this talk, \((M, \omega)\) is a closed symplectic manifold and \(S^1 = \mathbb{R}/\mathbb{Z}\).

\[
H : S^1 \times M \to \mathbb{R} \text{ smooth} \quad \leadsto \quad \text{time-dependent vector field } X_H,
\]

given by \(\iota_{X_H} \omega = d(H(t, \cdot))\)

\[
\leadsto \quad \text{Hamiltonian flow } \{\phi^t_H\}_{t \in \mathbb{R}},
\]

\[
\phi^0_H = \text{Id}, \quad \frac{d}{dt}(\phi^t_H(p)) = X_H(t, \phi^t_H(p)).
\]

\(H\) is called non-degenerate if, for each fixed point \(p \in M\) of \(\phi^1_H\), the linearization \((\phi^1_H)_*: T_pM \to T_pM\) has all eigenvalues different from 1.
Throughout this talk, \((M, \omega)\) is a closed symplectic manifold and \(S^1 = \mathbb{R}/\mathbb{Z}\).

\[
H : S^1 \times M \to \mathbb{R} \text{ smooth } \quad \leadsto \quad \text{time-dependent vector field } X_H,
\]

\[
\text{given by } \iota_{X_H} \omega = d(H(t, \cdot))
\]

\[
\leadsto \quad \text{Hamiltonian flow } \{\phi^t_H\}_{t \in \mathbb{R}}
\]

\[
\phi^0_H = Id, \quad \frac{d}{dt}(\phi^t_H(p)) = X_H(t, \phi^t_H(p)).
\]

\(H\) is called non-degenerate if, for each fixed point \(p \in M\) of \(\phi^1_H\), the linearization \((\phi^1_H)_* : T_pM \to T_pM\) has all eigenvalues different from 1.
Throughout this talk, \((M, \omega)\) is a closed symplectic manifold and 
\[ S^1 = \mathbb{R}/\mathbb{Z}. \]

\[ H : S^1 \times M \to \mathbb{R} \text{ smooth} \quad \mapsto \quad \text{time-dependent vector field } X_H, \]

\[ \text{given by } \iota_{X_H} \omega = d(H(t, \cdot)) \]

\[ \mapsto \quad \text{Hamiltonian flow } \{ \phi^t_H \}_{t \in \mathbb{R}} \]

\[ \phi^0_H = \text{Id}, \quad \frac{d}{dt} (\phi^t_H(p)) = X_H(t, \phi^t_H(p)). \]

\(H\) is called \textit{non-degenerate} if, for each fixed point \(p \in M\) of \(\phi^1_H\),
the linearization \((\phi^1_H)_* : T_pM \to T_pM\) has all eigenvalues different from 1.
Let
\[ L_0 M = \{ \text{contractible loops } \gamma: S^1 \to M \}. \]

Define a 1-form \( a_H \in \Omega^1(L_0 M) \) by
\[
(a_H)_\gamma(\xi) = \int_{S^1} \omega_{\gamma(t)}(\dot{\gamma}(t), \xi(t))dt - \int_{S^1} (dH)_{\gamma(t)}(\xi(t))dt.
\]

\( a_H \) vanishes at \( \gamma \) precisely if \( \gamma(t) = \phi^t_H(p) \) where \( p \in \text{Fix}(\phi^1_H) \).
Let
\[ \mathcal{L}_0M = \{ \text{contractible loops } \gamma : S^1 \to M \} .\]

Define a 1-form \( a_H \in \Omega^1(\mathcal{L}_0M) \) by
\[
(a_H)_\gamma(\xi) = \int_{S^1} \omega_{\gamma(t)}(\dot{\gamma}(t), \xi(t)) dt - \int_{S^1} (dH)_{\gamma(t)}(\dot{\xi}(t)) dt.
\]

\( a_H \) vanishes at \( \gamma \) precisely if \( \gamma(t) = \phi_H^t(p) \) where \( p \in Fix(\phi_H^1) \).
Let
\[ \mathcal{L}_0 M = \{ \text{contractible loops } \gamma: S^1 \to M \}. \]

Define a 1-form \( a_H \in \Omega^1(\mathcal{L}_0 M) \) by
\[
(a_H)_\gamma(\xi) = \int_{S^1} \omega_{\gamma(t)}(\dot{\gamma}(t), \xi(t)) dt - \int_{S^1} (dH)_{\gamma(t)}(\xi(t)) dt.
\]

\( a_H \) vanishes at \( \gamma \) precisely if \( \gamma(t) = \phi_H^t(p) \) where \( p \in \text{Fix}(\phi_H^1) \).
The 1-form $a_H$ on $\mathcal{L}_0M$ is closed; the pullback of $a_H$ to the cover

$$\tilde{\mathcal{L}}_0M = \{(\gamma, w) | \gamma \in \mathcal{L}_0M, w : D^2 \to M, w|_{\partial D^2} = \gamma\}$$

$(\gamma, w) \sim (\gamma', w')$ iff $\gamma = \gamma'$ and $\int_{D^2} w^* \omega = \int_{D^2} w'^* \omega$

is exact; specifically, the pullback is $dA_H$ where

$$A_H([\gamma, w]) = -\int_{D^2} w^* \omega - \int_{S^1} H(t, \gamma(t))dt.$$ 

Thus the set of critical points of $A_H$ comprises one orbit of the covering group of $\tilde{\mathcal{L}}_0M \to \mathcal{L}_0M$ for each fixed point of $\phi_H^1$ whose orbit under $\{\phi_H^t\}$ is contractible.

$A_H$ is a Morse function iff $H$ is nondegenerate, and Hamiltonian Floer homology is Morse-Novikov homology for

$$A_H : \tilde{\mathcal{L}}_0M \to \mathbb{R}.$$
The 1-form $a_H$ on $L_0 M$ is closed; the pullback of $a_H$ to the cover

$$\tilde{L}_0 M = \{(\gamma, w) | \gamma \in L_0 M, w: D^2 \to M, w|_{\partial D^2} = \gamma \}$$

is exact; specifically, the pullback is $dA_H$ where

$$A_H([\gamma, w]) = - \int_{D^2} w^* \omega - \int_{S^1} H(t, \gamma(t)) dt.$$

Thus the set of critical points of $A_H$ comprises one orbit of the covering group of $\tilde{L}_0 M \to L_0 M$ for each fixed point of $\phi^1_H$ whose orbit under $\{\phi^t_H\}$ is contractible.

$A_H$ is a Morse function iff $H$ is nondegenerate, and Hamiltonian Floer homology is Morse-Novikov homology for

$$A_H: \tilde{L}_0 M \to \mathbb{R}.$$
The 1-form $\alpha_H$ on $L_0M$ is closed; the pullback of $\alpha_H$ to the cover

$$\widetilde{L}_0M = \{(\gamma,w) | \gamma \in L_0M, w: D^2 \to M, w|_{\partial D^2} = \gamma\}$$

$(\gamma,w) \sim (\gamma',w')$ iff $\gamma = \gamma'$ and $\int_{D^2} w^* \omega = \int_{D^2} w'^* \omega$

is exact; specifically, the pullback is $dA_H$ where

$$A_H([\gamma, w]) = -\int_{D^2} w^* \omega - \int_{S^1} H(t, \gamma(t))dt.$$

Thus the set of critical points of $A_H$ comprises one orbit of the covering group of $\widetilde{L}_0M \to L_0M$ for each fixed point of $\phi_H^1$ whose orbit under $\{\phi_H^t\}$ is contractible.

$A_H$ is a Morse function iff $H$ is nondegenerate, and Hamiltonian Floer homology is Morse-Novikov homology for

$$A_H: \widetilde{L}_0M \to \mathbb{R}.$$
The 1-form $\alpha_H$ on $\mathcal{L}_0M$ is closed; the pullback of $\alpha_H$ to the cover

$$\widetilde{\mathcal{L}}_0M = \{(\gamma, w) | \gamma \in \mathcal{L}_0M, w: D^2 \to M, w|_{\partial D^2} = \gamma\}$$

is exact; specifically, the pullback is $d\mathcal{A}_H$ where

$$\mathcal{A}_H([\gamma, w]) = -\int_{D^2} w^*\omega - \int_{S^1} H(t, \gamma(t))dt.$$ 

Thus the set of critical points of $\mathcal{A}_H$ comprises one orbit of the covering group of $\widetilde{\mathcal{L}}_0M \to \mathcal{L}_0M$ for each fixed point of $\phi^1_H$ whose orbit under $\{\phi^t_H\}$ is contractible.

$\mathcal{A}_H$ is a Morse function iff $H$ is nondegenerate, and Hamiltonian Floer homology is Morse-Novikov homology for

$$\mathcal{A}_H: \widetilde{\mathcal{L}}_0M \to \mathbb{R}.$$
As a group, the Floer chain complex is

\[ CF_*(H) = \{ \sum c_i[\gamma_i, w_i] | c_i \in \mathbb{Q}, [\gamma_i, w_i] \in \text{Crit}(A_H), A_H([\gamma_i, w_i]) \downarrow -\infty \} \].

The boundary operator (which depends on auxiliary data, in particular on an almost complex structure \( J \)) counts negative gradient flowlines of \( A_H \):

\[
\partial [\gamma^-, w^-] = \sum n_{[\gamma^-, w^-],[\gamma^+, w^+]} [\gamma^+, w^+]
\]

where \( n_{[\gamma^-, w^-],[\gamma^+, w^+]} \) is a formal count of index-one solutions \( u: \mathbb{R} \times S^1 \to M \) to

\[
\frac{\partial u}{\partial s} + J(u(s, t)) \left( \frac{\partial u}{\partial t} - X_H(t, u(s, t)) \right) = 0
\]

such that \( u(s, \cdot) \to \gamma^\pm \) as \( s \to \pm \infty \) and \( [\gamma^+, w^+] = [\gamma^+, w^- \# u] \).
As a group, the Floer chain complex is

\[ CF_*(H) = \left\{ \sum c_i [\gamma_i, w_i] | c_i \in \mathbb{Q}, [\gamma_i, w_i] \in \text{Crit}(\mathcal{A}_H), \mathcal{A}_H([\gamma_i, w_i]) \downarrow -\infty \right\} \].

The boundary operator (which depends on auxiliary data, in particular on an almost complex structure \( J \)) counts negative gradient flowlines of \( \mathcal{A}_H \):

\[ \partial [\gamma^-, w^-] = \sum n_{[\gamma^-, w^-], [\gamma^+, w^+]} [\gamma^+, w^+] \]

where \( n_{[\gamma^-, w^-], [\gamma^+, w^+]} \) is a formal count of index-one solutions \( u: \mathbb{R} \times S^1 \to M \) to

\[ \frac{\partial u}{\partial s} + J(u(s, t)) \left( \frac{\partial u}{\partial t} - X_H(t, u(s, t)) \right) = 0 \]

such that \( u(s, \cdot) \to \gamma^\pm \) as \( s \to \pm \infty \) and \( [\gamma^+, w^+] = [\gamma^+, w^- \# u] \).
As a group, the Floer chain complex is

\[ CF_*(H) = \left\{ \sum c_i [\gamma_i, w_i] | c_i \in \mathbb{Q}, [\gamma_i, w_i] \in \text{Crit}(\mathcal{A}_H), \mathcal{A}_H([\gamma_i, w_i]) \downarrow -\infty \right\}. \]

The boundary operator (which depends on auxiliary data, in particular on an almost complex structure \( J \)) counts negative gradient flowlines of \( \mathcal{A}_H \):

\[
\partial [\gamma^-, w^-] = \sum n_{[\gamma^-, w^-], [\gamma^+, w^+]} [\gamma^+, w^+]
\]

where \( n_{[\gamma^-, w^-], [\gamma^+, w^+]} \) is a formal count of index-one solutions \( u: \mathbb{R} \times S^1 \to M \) to

\[
\frac{\partial u}{\partial s} + J(u(s, t)) \left( \frac{\partial u}{\partial t} - X_H(t, u(s, t)) \right) = 0
\]

such that \( u(s, \cdot) \to \gamma^\pm \) as \( s \to \pm \infty \) and \( [\gamma^+, w^+] = [\gamma^+, w^- \# u] \).
As a group, the Floer chain complex is

$$CF_*(H) = \left\{ \sum c_i [\gamma_i, w_i] \middle| c_i \in \mathbb{Q}, [\gamma_i, w_i] \in \text{Crit}(\mathcal{A}_H), \mathcal{A}_H([\gamma_i, w_i]) \searrow -\infty \right\}.$$ 

The boundary operator (which depends on auxiliary data, in particular on an almost complex structure $J$) counts negative gradient flowlines of $\mathcal{A}_H$:

$$\partial [\gamma^-, w^-] = \sum n_{[\gamma^-, w^-], [\gamma^+, w^+]} [\gamma^+, w^+]$$

where $n_{[\gamma^-, w^-], [\gamma^+, w^+]}$ is a formal count of index-one solutions $u: \mathbb{R} \times S^1 \to M$ to

$$\frac{\partial u}{\partial s} + J(u(s, t)) \left( \frac{\partial u}{\partial t} - X_H(t, u(s, t)) \right) = 0$$

such that $u(s, \cdot) \to \gamma^\pm$ as $s \to \pm \infty$ and $[\gamma^+, w^+] = [\gamma^+, w^- \# u]$. 
Theorem (Floer, Hofer-Salamon, Fukaya-Ono, Liu-Tian)

This can be carried out for a nondegenerate Hamiltonian $H$ on an arbitrary closed symplectic $(M, \omega)$ with coefficients in an appropriate (Novikov) ring $\Lambda$; one has $\partial^2 = 0$, and the resulting homology $HF_*(H)$ satisfies

$$HF_*(H) \cong H_*(M, \mathbb{Q}) \otimes \Lambda,$$

independently of $H$

Corollary (variant of Arnold’s conjecture)

If $H : S^1 \times M \to \mathbb{R}$ is nondegenerate then the number of fixed points of $\phi^1_H$ is at least the sum of the Betti numbers of $M$. 
Main theme of this talk: Although the Floer homology $HF_* (H)$ is independent of $H$, the underlying chain complex $CF_* (H)$ carries a $\mathbb{R}$-valued filtration, and this filtration carries interesting information that is specific to the isotopy $\{ \phi^t_H \}_{0 \leq t \leq 1}$.
Recall

$$\mathcal{A}_H([\gamma, w]) = -\int_{D^2} w^* \omega - \int_{S^1} H(t, \gamma(t)) dt$$

and

$$CF_*(H) = \left\{ \sum c_i [\gamma_i, w_i] \mid c_i \in \mathbb{Q}, [\gamma_i, w_i] \in \text{Crit}(\mathcal{A}_H), \mathcal{A}_H([\gamma_i, w_i]) \downarrow -\infty \right\}.$$ 

For

$$c = \sum c_i [\gamma_i, w_i] \in CF_*(H),$$

put

$$L_H(c) = \max_{c_i \neq 0} \mathcal{A}_H([\gamma_i, w_i]).$$

Then, for any $\lambda \in \mathbb{R}$, define

$$CF^\lambda_*(H) = \{ c \in CF_*(H) \mid L_H(c) \leq \lambda \}.$$
Recall
\[ \mathcal{A}_H([\gamma, w]) = -\int_{D^2} w^* \omega - \int_{S^1} H(t, \gamma(t)) dt \]
and
\[ CF_\ast(H) = \left\{ \sum c_i [\gamma_i, w_i] \mid c_i \in \mathbb{Q}, [\gamma_i, w_i] \in \text{Crit}(\mathcal{A}_H), \mathcal{A}_H([\gamma_i, w_i]) \downarrow -\infty \right\}. \]

For
\[ c = \sum c_i [\gamma_i, w_i] \in CF_\ast(H), \]
put
\[ \mathcal{L}_H(c) = \max_{c_i \neq 0} \mathcal{A}_H([\gamma_i, w_i]). \]

Then, for any \( \lambda \in \mathbb{R} \), define
\[ CF^\lambda_\ast(H) = \{ c \in CF_\ast(H) \mid \mathcal{L}_H(c) \leq \lambda \}. \]
Recall

\[ A_H([\gamma, w]) = -\int_{D^2} w^* \omega - \int_{S^1} H(t, \gamma(t)) dt \]

and

\[ CF_*(H) = \{ \sum c_i [\gamma_i, w_i] | c_i \in \mathbb{Q}, [\gamma_i, w_i] \in Crit(A_H), A_H([\gamma_i, w_i]) \downarrow -\infty \} \].

For

\[ c = \sum c_i [\gamma_i, w_i] \in CF_*(H), \]

put

\[ L_H(c) = \max_{c_i \neq 0} A_H([\gamma_i, w_i]). \]

Then, for any \( \lambda \in \mathbb{R} \), define

\[ CF^\lambda_*(H) = \{ c \in CF_*(H) | L_H(c) \leq \lambda \}. \]
\[ CF_\lambda^*(H) = \{ c \in CF_\ast(H) | \mathcal{L}_H(c) \leq \lambda \}. \]
\[ \mathcal{L}_H(c) = \max_{c_i \neq 0} \mathcal{A}_H([\gamma_i, w_i]). \]

Whenever \( u : \mathbb{R} \times S^1 \to M \) contributes to the matrix element \( n[\gamma^-, w^-], [\gamma^+, w^+] \) for the Floer boundary operator, one has
\[ \mathcal{A}_H([\gamma^-, w^-]) - \mathcal{A}_H([\gamma^+, w^+]) = \int \left| \frac{\partial u}{\partial s} \right|^2 > 0. \]

Hence
\[ \mathcal{L}_H(\partial c) < \mathcal{L}_H(c), \]
and in particular for any \( \lambda \in \mathbb{R} \) the \( \lambda \)-filtered part \( CF_\lambda^*(H) \) is preserved by the boundary operator.
Hamiltonian Floer homology

The filtration

The Oh-Schwarz spectral invariant

The boundary depth

\[ CF^\lambda_\ast (H) = \{ c \in CF_\ast (H) | \mathcal{L}_H (c) \leq \lambda \}. \]

\[ \mathcal{L}_H (c) = \max_{c_i \neq 0} \mathcal{A}_H ([\gamma_i, w_i]). \]

Whenever \( u : \mathbb{R} \times S^1 \rightarrow M \) contributes to the matrix element

\[ n[\gamma^-, w^-], [\gamma^+, w^+] \]

for the Floer boundary operator, one has

\[ \mathcal{A}_H ([\gamma^-, w^-]) - \mathcal{A}_H ([\gamma^+, w^+]) = \int \left| \frac{\partial u}{\partial s} \right|^2 > 0. \]

Hence

\[ \mathcal{L}_H (\partial c) < \mathcal{L}_H (c), \]

and in particular for any \( \lambda \in \mathbb{R} \) the \( \lambda \)-filtered part \( CF^\lambda_\ast (H) \) is preserved by the boundary operator.
Hamiltonian Floer homology

The filtration

The Oh-Schwarz spectral invariant

The boundary depth

\[ CF_\lambda^* (H) = \{ c \in CF_\ast (H) | \mathcal{L}_H (c) \leq \lambda \}. \]

\[ \mathcal{L}_H (c) = \max_{c_i \neq 0} \mathcal{A}_H ([\gamma_i, w_i]). \]

Whenever \( u : \mathbb{R} \times S^1 \rightarrow M \) contributes to the matrix element

\[ n[\gamma^-, w^-], [\gamma^+, w^+] \]

for the Floer boundary operator, one has

\[ \mathcal{A}_H ([\gamma^-, w^-]) - \mathcal{A}_H ([\gamma^+, w^+]) = \int \left| \frac{\partial u}{\partial s} \right|^2 > 0. \]

Hence

\[ \mathcal{L}_H (\partial c) < \mathcal{L}_H (c), \]

and in particular for any \( \lambda \in \mathbb{R} \) the \( \lambda \)-filtered part \( CF_\lambda^* (H) \) is preserved by the boundary operator.
Theorem

Given two normalized (\( \int_M H(t, \cdot) \omega^n = 0 \)) Hamiltonians \( H_0, H_1 \) such that \( \phi^1_{H_0} = \phi^1_{H_1} \) and the paths \( \{\phi^t_{H_i}\}_{0 \leq t \leq 1} \) are homotopic rel endpoints in the Hamiltonian diffeomorphism group, and given sets of auxiliary data needed to define the boundary operators on \( CF_*(H_i) \), there is an isomorphism of chain complexes

\[
\Phi: CF_*(H_0) \to CF_*(H_1)
\]

which, for each \( \lambda \in \mathbb{R} \), restricts to an isomorphism

\[
CF^\lambda_*(H_0) \to CF^\lambda_*(H_1).
\]

Furthermore, the induced isomorphism \( \Phi_*: HF_*(H_0) \to HF_*(H_1) \) commutes with the Piunikhin-Salamon-Schwarz isomorphisms \( \Psi_{H_i}: H_*(M; \mathbb{Q}) \otimes \Lambda \cong HF_*(H_i) \).
Theorem

Given two normalized ($\int_M H(t, \cdot) \omega^n = 0$) Hamiltonians $H_0, H_1$ such that $\phi_{H_0}^1 = \phi_{H_1}^1$ and the paths $\{\phi_{H_i}^t\}_{0 \leq t \leq 1}$ are homotopic rel endpoints in the Hamiltonian diffeomorphism group, and given sets of auxiliary data needed to define the boundary operators on $\text{CF}_*(H_i)$, there is an isomorphism of chain complexes

$$\Phi: \text{CF}_*(H_0) \rightarrow \text{CF}_*(H_1)$$

which, for each $\lambda \in \mathbb{R}$, restricts to an isomorphism

$$\text{CF}_*^\lambda (H_0) \rightarrow \text{CF}_*^\lambda (H_1).$$

Furthermore, the induced isomorphism $\Phi_*: \text{HF}_*(H_0) \rightarrow \text{HF}_*(H_1)$ commutes with the Piunikhin-Salamon-Schwarz isomorphisms $\Psi_{H_i}: H_* (M; \mathbb{Q}) \otimes \Lambda \cong \text{HF}_*(H_i)$. 
Thus the $\mathbb{R}$-filtered chain isomorphism type of the Floer chain complex is an invariant of the class of $\{\phi^t_H\}_{0 \leq t \leq 1}$ in $\widetilde{Ham}(M, \omega)$. Certain numerical invariants that can be extracted from this filtered chain isomorphism type have proven useful in Hamiltonian dynamics.
Thus the $\mathbb{R}$-filtered chain isomorphism type of the Floer chain complex is an invariant of the class of $\{\phi^t_H\}_{0 \leq t \leq 1}$ in $\tilde{\text{Ham}}(M, \omega)$. Certain numerical invariants that can be extracted from this filtered chain isomorphism type have proven useful in Hamiltonian dynamics.
Compare with the situation in Heegaard Floer homology: Given a null-homologous knot $K$ in a 3-manifold $Y$, one has a chain complex $\widehat{CF}(Y)$ whose chain homotopy type only depends on $Y$, but with a filtration that carries information about $K$.

Differences:

- In the Heegaard Floer case the filtration is by $\mathbb{Z}$, rather than $\mathbb{R}$.
- The Hamiltonian Floer differential *strictly* lowers the filtration level.
- In knot Floer homology only the filtered chain homotopy type is a knot invariant, whereas in Hamiltonian Floer theory the filtered chain isomorphism type is an invariant of the Hamiltonian.
Compare with the situation in Heegaard Floer homology: Given a null-homologous knot $K$ in a 3-manifold $Y$, one has a chain complex $\widehat{CF}(Y)$ whose chain homotopy type only depends on $Y$, but with a filtration that carries information about $K$.

Differences:

- In the Heegaard Floer case the filtration is by $\mathbb{Z}$, rather than $\mathbb{R}$.
- The Hamiltonian Floer differential strictly lowers the filtration level.
- In knot Floer homology only the filtered chain homotopy type is a knot invariant, whereas in Hamiltonian Floer theory the filtered chain isomorphism type is an invariant of the Hamiltonian.
We have the PSS isomorphism

\[ \Psi_H : H_*(M; \mathbb{Q}) \otimes \Lambda \rightarrow HF_*(H). \]

For

\[ a \in H_*(M; \mathbb{Q}) \otimes \Lambda, \]

put

\[ \rho(H;a) = \inf \{ \mathcal{L}_H(c) | c \text{ represents } \Psi_H(a) \in HF_*(H) \}. \]

This depends only on \((H, a)\) (and not on the auxiliary data used to define the Floer boundary operator), and extends continuously to all \(C^0\) functions \(H : S^1 \times M \rightarrow \mathbb{R}\) (rather than just smooth nondegenerate Hamiltonians).

(This is similar to the \(\tau\) invariant in Heegaard Floer theory.)
We have the PSS isomorphism
\[
\Psi_H : H_\ast(M; \mathbb{Q}) \otimes \Lambda \to HF_\ast(H).
\]

For \(a \in H_\ast(M; \mathbb{Q}) \otimes \Lambda\), put
\[
\rho(H; a) = \inf \{ \mathcal{L}_H(c) | c \text{ represents } \Psi_H(a) \in HF_\ast(H) \}.
\]

This depends only on \((H, a)\) (and not on the auxiliary data used to define the Floer boundary operator), and extends continuously to all \(C^0\) functions \(H : S^1 \times M \to \mathbb{R}\) (rather than just smooth nondegenerate Hamiltonians).
(This is similar to the \(\tau\) invariant in Heegaard Floer theory.)
We have the PSS isomorphism

$$\Psi_H: H_\ast(M; \mathbb{Q}) \otimes \Lambda \to HF_\ast(H).$$

For

$$a \in H_\ast(M; \mathbb{Q}) \otimes \Lambda,$$

put

$$\rho(H; a) = \inf\{ L_H(c) | c \text{ represents } \Psi_H(a) \in HF_\ast(H) \}.$$

This depends only on \((H, a)\) (and not on the auxiliary data used to define the Floer boundary operator), and extends continuously to all \(C^0\) functions \(H: S^1 \times M \to \mathbb{R}\) (rather than just smooth nondegenerate Hamiltonians).

(This is similar to the \(\tau\) invariant in Heegaard Floer theory.)
We have the PSS isomorphism

\[ \Psi_H: H_*(M; \mathbb{Q}) \otimes \Lambda \to HF_*(H). \]

For \( a \in H_*(M; \mathbb{Q}) \otimes \Lambda \),

put

\[ \rho(H; a) = \inf \{ L_H(c) | c \text{ represents } \Psi_H(a) \in HF_*(H) \}. \]

This depends only on \((H, a)\) (and not on the auxiliary data used to define the Floer boundary operator), and extends continuously to all \( C^0 \) functions \( H: S^1 \times M \to \mathbb{R} \) (rather than just smooth nondegenerate Hamiltonians).

(This is similar to the \( \tau \) invariant in Heegaard Floer theory.)
Example

If \( H(t, m) = f(m) \) for a sufficiently \( C^2 \)-small Morse function \( f: M \to \mathbb{R} \), then \((CF_*(H), \partial)\) coincides with the Thom-Smale-Morse-Witten complex \((CM_*(-f), \partial_{\text{Morse}})\) of the Morse function \(-f\).

Consequently, in this case, setting \( a = [M] \in H_*(M; \mathbb{Q}) \otimes \Lambda \),

\[
\rho(H; [M]) = \max_{S^1 \times M} (-H).
\]

Theorem (Oh, U.)

If \( H: S^1 \times M \to \mathbb{R} \) is autonomous (i.e., independent of the \( S^1 \)-variable), and if \( X_H \) has no nonconstant contractible periodic orbits of period \( \leq 1 \), then

\[
\rho(H; [M]) = \max_{S^1 \times M} (-H).
\]
Example

If $H(t, m) = f(m)$ for a sufficiently $C^2$-small Morse function $f : M \to \mathbb{R}$, then $(CF_*(H), \partial)$ coincides with the Thom-Smale-Morse-Witten complex $(CM_*(-f), \partial_{Morse})$ of the Morse function $-f$.

Consequently, in this case, setting $a = [M] \in H_*(M; \mathbb{Q}) \otimes \Lambda$,

$$\rho(H; [M]) = \max_{S^1 \times M} (-H).$$

Theorem (Oh, U.)

If $H : S^1 \times M \to \mathbb{R}$ is autonomous (i.e., independent of the $S^1$-variable), and if $X_H$ has no nonconstant contractible periodic orbits of period $\leq 1$, then

$$\rho(H; [M]) = \max_{S^1 \times M} (-H).$$
Example

If \( H(t, m) = f(m) \) for a sufficiently \( C^2 \)-small Morse function \( f : M \to \mathbb{R} \), then \( (CF_*(H), \partial) \) coincides with the Thom-Smale-Morse-Witten complex \( (CM_*(-f), \partial_{\text{Morse}}) \) of the Morse function \(-f\).
Consequently, in this case, setting \( a = [M] \in H_*(M; \mathbb{Q}) \otimes \Lambda \),

\[
\rho(H; [M]) = \max_{S^1 \times M} (-H).
\]

Theorem (Oh, U.)

If \( H : S^1 \times M \to \mathbb{R} \) is autonomous (i.e., independent of the \( S^1 \)-variable), and if \( X_H \) has no nonconstant contractible periodic orbits of period \( \leq 1 \), then

\[
\rho(H; [M]) = \max_{S^1 \times M} (-H).
\]
This allows one to use the spectral invariant to estimate the $(\pi_1$-sensitive) **Hofer-Zehnder capacity**: by definition, if $U \subset M$ is open,

$$c^\circ_{HZ}(U) = \sup \left\{ \max H \left| \begin{array}{c}
H \text{ is autonomous, } supp H \subset S^1 \times U, \\
\text{and } H \text{ has no nonconstant contractible periodic orbits of period } \leq 1
\end{array} \right. \right\}$$

Thus the previous theorem shows that

$$c^\circ_{HZ}(U) \leq \sup \{ \rho(H; [M]) | supp H \subset S^1 \times U \}$$
This allows one to use the spectral invariant to estimate the \((\pi_1\text{-sensitive})\) \textbf{Hofer-Zehnder capacity}: by definition, if \(U \subset M\) is open,

\[
c_{HZ}^o(U) = \sup \left\{ \max H \mid \begin{array}{l} H \text{ is autonomous, } \text{supp}H \subset S^1 \times U, \\
\text{and } H \text{ has no nonconstant contractible periodic orbits of period } \leq 1 \end{array} \right\}
\]

Thus the previous theorem shows that

\[
c_{HZ}^o(U) \leq \sup \{ \rho(H; [M]) \mid \text{supp}H \subset S^1 \times U \}
\]
Meanwhile:

**Theorem (Frauenfelder-Ginzburg-Schlenk, U.)**

*If* \( L \subset M \) *is compact,* \( \text{supp} H \subset S^1 \times L \), *and* \( \phi_K^1(L) \cap L = \emptyset \), *then*

\[
\rho(H; [M]) \leq \int_0^1 \left( \max_M K(t, \cdot) - \min_M K(t, \cdot) \right) dt =: \| K \|.
\]

Where the *displacement energy* is defined by

\[
e(L, M) = \inf\{ \| K \| \mid \phi_K^1(L) \cap L = \emptyset \}
\]

for \( L \) *compact and*

\[
e(S, M) = \sup_{L \in S} e(L, M)
\]

in general, it follows that:
Theorem (U.)

For any subset $S \subset M$ (where $M$ is any closed symplectic manifold), we have

$$c^\circ_{HZ}(S) \leq e(S, M).$$

Hofer–Zehnder proved this for $M = \mathbb{R}^{2n}$ in the early '90s, but for general $M$ it had only been proven up to a constant. The result is sharp: for $S$ equal to a small Darboux ball $B^{2n}(r)$ one has $c^\circ_{HZ}(S) = e(S, M) = \pi r^2$.

Sample non-squeezing consequence: If $\Sigma$ is any (possibly very low-area) closed surface, $N$ any closed or Stein 4-manifold, and $r < R$, there is no symplectic embedding $B^4(R) \times \Sigma \hookrightarrow B^2(r) \times N$. 
Theorem (U.)

For any subset $S \subset M$ (where $M$ is any closed symplectic manifold), we have

$$c_{HZ}^\circ(S) \leq e(S, M).$$

Hofer–Zehnder proved this for $M = \mathbb{R}^{2n}$ in the early ’90s, but for general $M$ it had only been proven up to a constant. The result is sharp: for $S$ equal to a small Darboux ball $B^{2n}(r)$ one has $c_{HZ}^\circ(S) = e(S, M) = \pi r^2$.

Sample non-squeezing consequence: If $\Sigma$ is any (possibly very low-area) closed surface, $N$ any closed or Stein 4-manifold, and $r < R$, there is no symplectic embedding $B^4(R) \times \Sigma \hookrightarrow B^2(r) \times N$. 
Theorem (U.)

For any subset $S \subset M$ (where $M$ is any closed symplectic manifold), we have

$$c_{HZ}^\circ(S) \leq e(S, M).$$

Hofer–Zehnder proved this for $M = \mathbb{R}^{2n}$ in the early ’90s, but for general $M$ it had only been proven up to a constant. The result is sharp: for $S$ equal to a small Darboux ball $B^{2n}(r)$ one has $c_{HZ}^\circ(S) = e(S, M) = \pi r^2$.

Sample non-squeezing consequence: If $\Sigma$ is any (possibly very low-area) closed surface, $N$ any closed or Stein 4-manifold, and $r < R$, there is no symplectic embedding $B^4(R) \times \Sigma \hookrightarrow B^2(r) \times N$. 
Definition

The boundary depth of a nondegenerate Hamiltonian $H$ on $M$ is

$$\beta(H) = \inf \{ \beta \geq 0 | (\forall \lambda > 0) (CF_*^\lambda(H) \cap \partial(CF_*^\lambda(H)) \subset \partial(CF_*^{\lambda+\beta}(H)) \}.$$ 

Non-obviously, $\beta(H)$ is finite (U. '07); in fact one has (Oh, ’07)

$$\beta(H) \leq \|H\|.$$ 

Theorem (U.)

(i) $\|\beta(H) - \beta(K)\| \leq \|H - K\|$; hence $\beta$ extends continuously to all continuous $H: S^1 \times M \to \mathbb{R}$ (and not just nondegenerate Hamiltonians)

(ii) If $H \leq 0$, $\text{supp}H \subset S^1 \times L$, and $\phi_K^1(L) \cap L = \emptyset$, then

$$\beta(H) \leq 2\|K\|. \quad (1)$$
**Definition**

The **boundary depth** of a nondegenerate Hamiltonian $H$ on $M$ is

$$\beta(H) = \inf \{ \beta \geq 0 | (\forall \lambda > 0) (CF_\lambda^*(H) \cap \partial(CF_*^*(H)) \subset \partial \left(CF_\lambda^* + \beta \right)^*(H)) \}.$$ 

Non-obviously, $\beta(H)$ is finite (U. ’07); in fact one has (Oh, ’07)

$$\beta(H) \leq \|H\|.$$ 

**Theorem (U.)**

(i) $\|\beta(H) - \beta(K)\| \leq \|H - K\|$; hence $\beta$ extends continuously to all continuous $H: S^1 \times M \to \mathbb{R}$ (and not just nondegenerate Hamiltonians)

(ii) If $H \leq 0$, supp$H \subset S^1 \times L$, and $\phi_K^1(L) \cap L = \emptyset$, then

$$\beta(H) \leq 2\|K\|.$$  (1)
Definition

The **boundary depth** of a nondegenerate Hamiltonian $H$ on $M$ is

$$
\beta(H) = \inf \{ \beta \geq 0 | (\forall \lambda > 0) (CF^\lambda_*(H) \cap \partial(CF_*(H)) \subset \partial \left( CF^\lambda_*(+\beta(H)) \right) \}.
$$

Non-obviously, $\beta(H)$ is finite (U. ’07); in fact one has (Oh, ’07)

$$
\beta(H) \leq \|H\|.
$$

Theorem (U.)

(i) $\|\beta(H) - \beta(K)\| \leq \|H - K\|; \text{ hence } \beta \text{ extends continuously to all continuous } H: S^1 \times M \to \mathbb{R} \text{ (and not just nondegenerate Hamiltonians)}$

(ii) If $H \leq 0$, supp$H \subset S^1 \times L$, and $\phi^1_K(L) \cap L = \emptyset$, then

$$
\beta(H) \leq 2\|K\|. \quad (1)
$$
Definition

The **boundary depth** of a nondegenerate Hamiltonian \( H \) on \( M \) is

\[
\beta(H) = \inf \{ \beta \geq 0 | (\forall \lambda > 0) (\text{CF}_*^\lambda(H) \cap \partial(\text{CF}_*^\lambda(H)) \subset \partial \left( \text{CF}_*^{\lambda+\beta}(H) \right) \}.
\]

Non-obviously, \( \beta(H) \) is finite (U. ’07); in fact one has (Oh, ’07)

\[
\beta(H) \leq \|H\|.
\]

Theorem (U.)

(i) \( \|\beta(H) - \beta(K)\| \leq \|H - K\|; \) hence \( \beta \) extends continuously to all continuous \( H: S^1 \times M \to \mathbb{R} \) (and not just nondegenerate Hamiltonians)

(ii) If \( H \leq 0, \text{supp}H \subset S^1 \times L, \) and \( \phi^1_K(L) \cap L = \emptyset, \) then

\[
\beta(H) \leq 2\|K\|. \tag{1}
\]
Some applications can be obtained by combining bounds on $\beta$ such as (1) with properties of the spectral invariants in order to deduce the existence of \textbf{low-energy} solutions $u: \mathbb{R} \times S^1 \to M$ to
\[ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_H) = 0 \]
having certain asymptotics.

A compactness argument allows one to show that this still works for degenerate $H$; one then deduces that if $H$ is supported in a suitably small set the resulting $u$ will be localized near this set.
Some applications can be obtained by combining bounds on $\beta$ such as (1) with properties of the spectral invariants in order to deduce the existence of low-energy solutions $u: \mathbb{R} \times S^1 \to M$ to $\frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_H) = 0$ having certain asymptotics. A compactness argument allows one to show that this still works for degenerate $H$; one then deduces that if $H$ is supported in a suitably small set the resulting $u$ will be localized near this set.
Applications of the boundary depth:

**Theorem (U., generalizing Schwarz, et al.)**

If \( e_{\text{stable}}(S, M) = 0 \) (e.g., if \( S \) is a non-Lagrangian submanifold with \( \dim S \leq \frac{1}{2} \dim M \), or if \( S \) is a symplectic submanifold) and if \( \langle [\omega], \pi_2(S) \rangle = 0 \), then there is a neighborhood \( W \) of \( S \) such that if \( H: S^1 \times M \to \mathbb{R} \) is a Hamiltonian with \( \text{supp} H \subset S^1 \times W \) and satisfying a technical condition, then \( \phi^1_H \) has infinitely many geometrically distinct nontrivial periodic points.

**Theorem (U., generalizing Ginzburg, Kerman)**

If \( N \subset M \) is a stable coisotropic submanifold, and if \( \langle [\omega], \pi_2(N) \rangle \) is discrete, then \( e(N, M) > 0 \). In particular, any coisotropic submanifold of contact type (in the sense of Bolle) has positive displacement energy.
Applications of the boundary depth:

**Theorem (U., generalizing Schwarz, et al.)**

If $e_{\text{stable}}(S, M) = 0$ (e.g., if $S$ is a non-Lagrangian submanifold with $\dim S \leq \frac{1}{2} \dim M$, or if $S$ is a symplectic submanifold) and if $\langle [\omega], \pi_2(S) \rangle = 0$, then there is a neighborhood $W$ of $S$ such that if $H: S^1 \times M \to \mathbb{R}$ is a Hamiltonian with $\text{supp} H \subseteq S^1 \times W$ and satisfying a technical condition, then $\phi_H^{1}$ has infinitely many geometrically distinct nontrivial periodic points.

**Theorem (U., generalizing Ginzburg, Kerman)**

If $N \subset M$ is a stable coisotropic submanifold, and if $\langle [\omega], \pi_2(N) \rangle$ is discrete, then $e(N, M) > 0$. In particular, any coisotropic submanifold of contact type (in the sense of Bolle) has positive displacement energy.