Periodic Localization, Tate Cohomology, and Infinite Loopspaces

Talk 1

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Introduction

Three talks

My goal is to introduce a circle of ideas and results involving

- Localization with respect to periodic homology theories.
- Infinite loopspace theory.
- The Tate construction in equivariant stable homotopy theory.

The topics fit with the functor calculus theme of the conference.
In more detail . . .

- **Talk 1**
  - Periodicity in stable homotopy.
  - Bousfield localization.
  - Telescopic functors associated to Morava $K$–theories.

- **Talk 2**
  - Homotopy orbits and fixed points.
  - The norm map and the Tate spectrum of a $G$–spectrum.
  - Vanishing results for Tate spectra after periodic localization.

- **Talk 3**
  - Applications to splitting Goodwillie towers.
  - Application to computing $E^n_*(\Omega^\infty X)$.
  - Open questions and speculation (after Arone-Ching).

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**The stable and unstable worlds**

$\mathcal{T} =$ the category of based topological spaces.

$ho(\mathcal{T}) =$ its homotopy category: weak equivalences have been inverted.

$\mathcal{S} =$ the category of spectra. In its basic flavor . . .

**Definition** A *spectrum* $X$ is a sequence of based spaces $X_0, X_1, \ldots$, together with maps $\Sigma X_d \to X_{d+1}$.

Homotopy groups $\ldots \pi_*(X) = \colim_d \pi_{*+d}(X_d)$.

Homology groups $\ldots E_*(X) = \colim_d E_{*+d}(X_d)$.

Weak equivalences: $f$ such that $\pi_*(f)$ is an iso. Invert these . . .

$ho(\mathcal{S}) =$ the associated homotopy category.
Why we like spectra

- $\text{ho}(S)$ is much more algebraic than $\text{ho}(T)$ . . .
  - It is triangulated.
  - Cofibration sequences are equivalent to fibration sequences.

- Every spectrum is naturally equivalent to an $\Omega$–spectrum, $X$ such that each $X_d \to \Omega X_{d+1}$ is a weak equivalence of spaces, and $\Omega$–spectra represent cohomology theories.

- $S$ is home for many of our friends: Thom spectra, $K$–theories, elliptic spectra, topological modular forms, . . .

Back and forth: suspension spectra and infinite loopspaces

**Definition** For $Z \in T$, $\Sigma^\infty Z \in S$ has $d$th space $\Sigma^d Z$, with identity structure maps.

**Definition** For $X \in S$, $\Omega^\infty X \in T$ is the 0th space of an $\Omega$–spectrum weakly equivalent to $X$.

The pair $\xymatrix{T \ar[r]^-{\Sigma^\infty} & S \ar[l]_-{\Omega^\infty}}$ induces adjoint functors on $\text{ho}(T)$.

$E_*(\Sigma^\infty Z) \simeq E_*(Z)$; $\Sigma^\infty$ preserves cofiber sequences, hocolimits, . . .

$\pi_*(\Omega^\infty Z) \simeq \pi_*(Z)$ for $* \geq 0$; $\Omega^\infty$ preserves fib. sequences, holims, . . .
An unstable/stable hybrid: periodic unstable homotopy

Computing homotopy groups $\pi_*(Z)$ is hard.

**Example** $\pi_*(S^3) = ?$ Generalize a hard problem . . .

**Definition** If $F$ is a finite complex, $\pi_n(Z; F) = [\Sigma^n F, Z]$

**Example** Computing $\pi_*(S^3; \mathbb{R}P^2)$ is still hard.

Simplify with localization . . . a self map $v : \Sigma^d F \to F$ (almost) makes $\pi_*(Z; F)$ a $\mathbb{Z}[v]$–module: given $f \in \pi_n(Z; F)$, $v \cdot f \in \pi_{n+d}(Z; F)$ is

$$\Sigma^{n+d} F \xrightarrow{v} \Sigma^n F \xrightarrow{f} Z.$$ 

Localize: $v^{-1} \pi_*(Z; F)$ is $\pi_*$ of the telescope of

$$\text{Map}_T(F, Z) \xrightarrow{v^*} \text{Map}_T(\Sigma^d F, Z) \xrightarrow{v^*} \text{Map}_T(\Sigma^{2d} F, Z) \xrightarrow{v^*} \ldots$$

or

$$\text{Map}_T(F, Z) \xrightarrow{v^*} \Omega^d \text{Map}_T(F, Z) \xrightarrow{v^*} \Omega^{2d} \text{Map}_T(F, Z) \xrightarrow{v^*} \ldots$$

Morava $K$–theories

Localized at a prime $p$, there exist $p$–local spectra

$$K(1), K(2), K(3), \ldots$$

- $K(n)$ is a complex oriented ring spectrum.
- $K(n)_* = \mathbb{Z}/p[\nu_n, \nu_n^{-1}], |\nu_n| = 2p^n - 2$.
- $K(n)_*(X \wedge Y) \simeq K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y)$.

$K(1)$ is (essentially) complex $K$–theory with mod $p$ coefficients.

It’s handy to define $K(0) = H\mathbb{Q}$. 
Morava E–theories

Localized at a prime $p$, there exist $p$–local spectra

$$E_1, E_2, E_3, \ldots$$

- $E_n$ is a complex oriented commutative $S$–algebra (a ring spectrum with a very nice multiplication).
- $E_{n*} = W(\mathbb{F}_{p^n})[u_1, \ldots, u_{n-1}][u, u^{-1}], |u_i| = 0, |u| = 2.$

$\mathbb{F}_{p^n}$ is the field with $p^n$ elements, $W(\mathbb{F}_{p^n}) \simeq \mathbb{Z}_p^n$ is its Witt ring.

$E_1$ is complex $K$–theory with $p$–adic coefficients.

$E_n$ should be viewed as an ‘integral lift’ of $K(n)$.

The chromatic filtration of finite spectra

Subcategories of $ho(S) \ldots$

$\mathcal{C} = p$–local finite CW spectra.

$\mathcal{C}_n = K(n-1)_*\text{–acyclics in } \mathcal{C}.$

**Theorem** The categories $\mathcal{C}_n$ are properly nested:

$$\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \mathcal{C}_2 \supset \ldots.$$ Proper: (Mitchell, 1983).

**Definition** An object $F \in \mathcal{C}_n - \mathcal{C}_{n+1}$ is said to be of type $n$. 
Periodicity in stable homotopy

The Nilpotence Theorem


Nilpotence Theorem  Given \( F \in C \),
\[
\nu : \Sigma^d F \to F \text{ is nilpotent } \iff K(n)_{\ast}(\nu) \text{ is nilpotent for all } n \geq 0.
\]

‘\( \nu \) is nilpotent’ means that there exists \( k \) such that the composite
\[
\Sigma^{kd} F \xrightarrow{\nu} \ldots \xrightarrow{\nu} \Sigma^{2d} F \xrightarrow{\nu} \Sigma^d F \xrightarrow{\nu} F
\]
is null.

A categorical characterization of the \( C_n \)'s . . .

Thick Subcategory Theorem  A nonempty full subcategory of \( C \) that is closed under taking cofibers and retracts is \( C_n \) for some \( n \).

\[ v_n \text{-self maps} \]

Definition  Given \( F \in C \), \( \nu : \Sigma^d F \to F \) is a \( v_n \)-self map if
\begin{itemize}
  \item \( K(n)_{\ast}(\nu) \) is an isomorphism.
  \item \( K(m)_{\ast}(\nu) \) is nilpotent for all \( m \neq n \).
\end{itemize}

Example  \( p : S^0 \to S^0 \) is a \( v_0 \)-self map.

Example (Adams) There exists a stable map \( A : \Sigma^8 \mathbb{R}P^2 \to \mathbb{R}P^2 \) that is multiplication by \( v_1^4 \) in \( K \)-theory. \( A \) is a \( v_1 \)-self map.

Remark  The cofiber of a \( v_n \)-self map will be in \( C_{n+1} \).
Periodicity in stable homotopy

Every kid wants an ice cream cone

Periodicity Theorem

- \( F \in C_n \iff F \) has a \( \nu_n \)-self map.

- Given \( F, G \in C_n \) with \( \nu_n \)-self maps \( u : \Sigma^c F \rightarrow F \) and \( v : \Sigma^d G \rightarrow G \), and \( f : F \rightarrow G \), there exist \( i, j \) such that \( ic = jd \) and the diagram

\[
\begin{array}{ccc}
\Sigma^i F & \xrightarrow{\Sigma^j f} & \Sigma^j G \\
\downarrow{u^i} & & \downarrow{v^j} \\
F & \xrightarrow{f} & G
\end{array}
\]

homotopy commutes.

Telescopes

For \( F \) of type \( n \), let \( T(F) \) be the mapping telescope of a \( \nu_n \)-self map:

\[
T(F) = \operatorname{hocolim}\{ F \xrightarrow{\nu} \Sigma^{-d} F \xrightarrow{\nu} \Sigma^{-2d} F \xrightarrow{\nu} \ldots \}.
\]

Consequences of the Periodicity Theorem . . .

- \( T(F) \) is independent of choice of self map.

- If \( F \) and \( F' \) are both of type \( n \), then

\[
T(F)_*(W) = 0 \iff T(F')_*(W) = 0.
\]

Definition Let \( T(n) \) ambiguously denote \( T(F) \) for any particular type \( n \) finite spectrum \( F \).

- \( T(n)_*(W) = 0 \Rightarrow K(n)_*(W) = 0. \)

Telescope Conjecture The converse is true. Still open for \( n > 1 \).
Resolutions

Another consequence of the Periodicity Theorem . . .

Resolution Theorem There exists a diagram in \( \mathcal{C} \),

\[
\begin{array}{cccccc}
F(1) & \xrightarrow{f(1)} & F(2) & \xrightarrow{f(2)} & F(3) & \rightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
S^0 & \swarrow & f(2) & \rightarrow & F(3) & \leftarrow \cdots \\
\end{array}
\]

such that each \( F(k) \in \mathcal{C}_n \), and \( \operatorname{hocolim}_k F(k) \rightarrow S^0 \) induces an \( T(m)_* \)-isomorphism for all \( m \geq n \).

Bousfield localization

\( E \)-local spectra

Definitions Fix a spectrum \( E \).

\( f : Y \rightarrow Z \) is an \( E_* \)-iso if \( E_*(f) \) is an isomorphism.

\( X \) is \( E \)-local if every \( E_* \)-iso \( f : Y \rightarrow Z \) induces a weak equivalence

\[
f^* : \operatorname{Map}_S(Z, X) \rightarrow \operatorname{Map}_S(Y, X).
\]

Remark This condition can be stated in terms of \( E_* \)-acyclics . . .

\[
X \text{ is } E \text{-local} \iff X^*(W) = 0 \text{ whenever } E_*(W) = 0 \\
\iff [W, X] = 0 \text{ whenever } E_*(W) = 0.
\]

Example \( K(n)_*(W) = 0 \Rightarrow K(n)^*(W) = 0 \Rightarrow E_n^*(W) = 0 \).
Thus \( E_n \) is \( K(n) \)-local.
Localization functors

(Bousfield, 1970’s) Given $E \in S$, there exists

- A functor $L_E : S \to S$.
- A natural transformation $\eta_X : X \to L_E X$ satisfying
  - $L_E X$ is $E$–local.
  - $\eta_X : X \to L_E X$ is an $E_*$–iso.

A formal consequence . . . $L_E$ is idempotent . . .

- $\eta_{L_E X} \simeq L(\eta_X) : L_E X \simeq L_EL_E X$.

$L_E$ inverts $E_*$–isos, and ‘kills’ $E_*$–acyclic spectra, in a minimal way.

Examples

Let $SG =$ Moore spectrum of type $G$.

Examples $L_{S\mathbb{Z}(p)} =$ localization at $p$. $L_{S\mathbb{Z}/p} =$ completion at $p$.

$L_E = L_F$ when $E_*(W) = 0 \iff F_*(W) = 0$.

Examples $L_{S\mathbb{Z}/p} = L_{S\mathbb{Z}/p^2}$. $L_K = LKO$.

More generally, $E_*(W) = 0 \Rightarrow F_*(W) = 0$ implies that $L_F X \simeq L_F L_E X$.

Example $L_{K(n)} = L_{K(n)}L_{T(n)}$.

Remark Telescope Conjecture asks if $L_{T(n)} = L_{K(n)}$. 
More examples

Constructions yielding $E$–local objects . . .

- $X \to Y \to Z$ a fib. seq. with 2 out of 3 $E$–local $\Rightarrow$ the 3rd is $E$–local.
- $X$ is $E$–local $\Rightarrow \text{Map}_S(Y, X)$ is $E$–local.
- $X_i, i \in I$, are $E$–local $\Rightarrow \prod_{i \in I} X_i$ is $E$–local.
- $G$ acts on an $E$–local $X \Rightarrow X^hG$ is $E$–local.
- $R$ a ring spectrum $\Rightarrow R$–module spectra (including $R$!) are $R$–local.

**Example** Completed at 2, $L_{K(1)}S$ is the fiber of $KO \xrightarrow{\Psi^3-1} KO$. Similar for $p$ odd. Note: $KO = K^h\mathbb{Z}/2$. There are now nice descriptions of $L_{K(2)}S$ in terms of spectra of the form $E^hG_2$ with $G$ finite. These are computationally useful!

Telescopic functors

The Periodicity Theorem well packaged for infinite loopspace theory . . .

**Theorem** (Bousfield $n = 1$, K. all $n$, 1980’s) For each $n \geq 1$ (and each $p$), there is a functor $\Phi_n : \mathcal{T} \to \mathcal{S}$ satisfying the following properties.

- For all spaces $Z$, $\Phi_n(Z)$ is $T(n)$–local.
- There is a natural isomorphism
  $$v^{-1}\pi_*(Z; F) \simeq [F, \Phi_n(Z)]_*$$
  for all unstable $v_n$–maps $v : \Sigma^dF \to F$, and spaces $Z$.
- For all spectra $X$, there is a natural weak equivalence
  $$\Phi_n(\Omega^\infty X) \simeq L_{T(n)}X.$$

**Remark** The first two properties (almost) characterize $\Phi_n$. 
Applications to spectra

Different spectra can have homotopy equivalent 0th spaces. However . . .

**Proposition** $\Omega^\infty X \simeq \Omega^\infty Y \Rightarrow L_{T(n)}X \simeq L_{T(n)}Y$ for all $n$.

**Proof:** Apply $\Phi_n$ to the equivalence $\Omega^\infty X \simeq \Omega^\infty Y$.

**Example** If $R$ is a commutative $S$–algebra, $\Omega_1^\infty R$ has a delooping classically denoted by $R_\otimes$, and more recently by $gl_1(R)$. So

$$L_{T(n)}gl_1(R) \simeq L_{T(n)}R.$$  

(Bousfield, 1982): With $n = 1$, recover the Adams–Priddy Thm that $BO_\otimes \simeq BO_\otimes$, suitably completed.

(Rezk, 2006): Let $R = E_n$, then identify, in terms of formal groups, the resulting ‘logarithm’ $l_{n,p} : E_0^0(Z)^\times \to E_0^0(Z)$.

Remark $\eta_n$ will make an appearance in the other talks.

Applications to spectra (cont.)

**Proposition** After $T(n)$–localization, the evaluation map

$$\epsilon : \Sigma^\infty \Omega^\infty X \to X$$

has a natural section

$$\eta_n : L_{T(n)}X \to L_{T(n)}\Sigma^\infty \Omega^\infty X.$$  

**Proof:** Apply $\Phi_n$ to the inclusion $(\Omega^\infty X) \to \Omega^\infty \Sigma^\infty (\Omega^\infty X)$.

**Corollary** For all $X$, $\epsilon^* : E_n^*(X) \to E_n^*(\Omega^\infty X)$ is split monic.

**Remark** By contrast, the kernel of $\epsilon^* : H^*(X; \mathbb{Z}/2) \to H^*(\Omega^\infty X; \mathbb{Z}/2)$ contains all elements of the form $Sq^i x$, $i > |x|$.  

**Remark** $\eta_n$ will make an appearance in the other talks.
Applications to spectra (cont.)

One more example, with $p = 2 \ldots$ there is a cofibration sequence

$$S^{-1} \xrightarrow{i} \mathbb{R}P_{-1}^\infty \xrightarrow{p} \mathbb{R}P_0^\infty \xrightarrow{t} S^0.$$ 

$i$ is the inclusion of the bottom cell. $t$ is the transfer map.

Kahn–Priddy: $\Omega^\infty t$ has a section. (Not quite true, but close enough.)

Thus $L_{T(n)} t$ has a section. We deduce that, localized at 2,

- $L_{K(n)} i$ is null for all $n$.
- $L_{K(n)} \mathbb{R}P_0^\infty \simeq L_{K(n)} (\mathbb{R}P_{-1}^\infty \vee S^0)$ for all $n$.

Contrast with speculation by Hopkins, Hovey \ldots

Conjecture $X, Y$ finite and $L_{K(n)} X \simeq L_{K(n)} Y$ for all $n \Rightarrow X \simeq Y$.

Applications to spaces

Roughly put, the spectrum $\Phi_n(Z)$ determines the unstable $v_n$–periodic homotopy groups of a space $Z$.

Problem For ones favorite $Z$, identify $\Phi_n(Z)$ in familiar terms.

More useful properties of $\Phi_n$:

- $\Phi_n$ takes homotopy pullbacks in $\mathcal{T}$ to homotopy pullbacks in $\mathcal{S}$.
- $\Phi_n(\text{Map}_\mathcal{T}(A, Z)) \simeq \text{Map}_\mathcal{S}(A, \Phi_n(Z))$ for all $A, Z \in \mathcal{T}$.
- $\Phi_n(Z \langle d \rangle) \simeq \Phi_n(Z)$ for all $Z \in \mathcal{T}$ and all $d$. 
Applications to spaces (cont.)

A strategy . . . ‘resolve’ the space \(Z\) by infinite loopspaces, and apply \(\Phi_n\).

Mahowald (1980): At 2, the James–Hopf map

\[
\Omega^{2m+1}S^{2m+1} \to \Omega\Sigma\Sigma\mathbb{R}P^{2m}
\]

induces an isomorphism on \(v_1\)-periodic homotopy.

Apply \(\Phi_1\) and deduce

**Theorem** Localized at 2, \(\Phi_1(S^{2m+1}) \simeq L_{K(1)}\Sigma^{2m+1}\mathbb{R}P^{2m}\).

R. Thompson: the odd primary analogue.

Applications to spaces (cont.)

A ‘conceptual’ proof, and generalization to higher \(n\) . . .

Apply \(\Phi_n\) to the resolution of \(S^{2m+1}\) by infinite loopspaces arising from the Goodwillie–Weiss tower of the identity, as analyzed by Arone–Mahowald.

(Help from B. Johnson and Dwyer.)

**Theorem** \(\Phi_n(S^{2m+1})\) has a finite resolution with fibers the \(T(n)\)-localization of known suspension spectra.

**Example** Localized at 2, there is a cofibration sequence

\[
\Phi_2(S^3) \to L_{T(2)}\Sigma^3\mathbb{R}P^2 \to L_{T(2)}\Sigma^3B,
\]

with \(B\) both \(K(1)_*\)-acyclic and fitting into a cofibration sequence

\[
\Sigma\mathbb{R}P^\infty / \mathbb{R}P^2 \to B \to \Sigma^2\mathbb{R}P^\infty / \mathbb{R}P^4.
\]
**Telescopic functors**

**Construction of $\Phi_n$**

**Step 1** An unstable map $\nu : \Sigma^d F \to F$ induces natural maps

\[
\text{Map}_T(F, Z) \xrightarrow{\nu^*} \Omega^d \text{Map}_T(F, Z) \xrightarrow{\nu^*} \Omega^{2d} \text{Map}_T(F, Z) \to \ldots
\]

Not interesting if $\nu$ is nilpotent. But if $\nu$ is $\nu_n$-periodic, define $\Phi_F : T \to S$ by letting $\Phi_F(Z)$ have $rd$th space $\text{Map}_T(F, Z)$, and structure maps

$$\Phi_F(Z)_{rd} \xrightarrow{\nu^*} \Omega^d \Phi_F(Z)_{(r+1)d}.$$ 

By periodicity, this extends to stable $F$, and is natural in $F$.

$\Phi_F$ satisfies versions of the properties of $\Phi_n$:

- $\Phi_F(Z)$ is $T(n)$–local.
- $\pi_*(\Phi_F(Z)) = \nu^{-1}\pi_*(Z; F)$.
- $\Phi_F(\Omega^\infty X) \simeq \text{Map}_S(F, L_{T(n)}X)$.

**Step 2** Recall that the Resolution Theorem says that one has

\[
\begin{array}{ccc}
F(1) & \xrightarrow{f(1)} & F(2) & \xrightarrow{f(2)} & F(3) & \to \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^0 & \to & \to & \to & \to & \to & \to
\end{array}
\]

such that each $F(k)$ is of type $n$, and $\text{hocolim}_{k} F(k) \to S^0$ is a $T(n)_*$–iso.

**Definition** $\Phi_n(Z) = \text{holim}_{k} \Phi_F(k)(Z)$.

Then $\ldots$ $\Phi_n(\Omega^\infty X) = \text{holim}_{k} \Phi_F(k)(\Omega^\infty X)$

\[\simeq \text{holim}_{k} \text{Map}_S(F(k), L_{T(n)}X)\]

\[= \text{Map}_S(\text{hocolim}_{k} F(k), L_{T(n)}X)\]

\[\simeq \text{Map}_S(S^0, L_{T(n)}X) = L_{T(n)}X.\]
Some References