Some Applications of Heegaard Floer homology to Dehn surgery

Yi Ni

American Institute of Mathematics
and
Massachusetts Institute of Technology

May 28, 2009
2009 Georgia Topology Conference
University of Georgia, Athens, GA
- Heegaard Floer homology
- Dehn surgery
- Cosmetic surgeries
- Property G
- Heegaard Floer homology
- Dehn surgery
- Cosmetic surgeries
- Property G

Some Applications of Heegaard Floer homology to Dehn surgery
Heegaard Floer homology was introduced by Ozsváth and Szabó in 2000. Using Heegaard diagrams and symplectic geometry, this theory assigns a package of invariants

$$\widehat{HF}(Y), \ HF^+(Y), \ HF^-(Y), \ HF^\infty(Y), \ldots$$

to a closed, oriented 3–manifold $Y$. 

Some Applications of Heegaard Floer homology to Dehn surgery
Heegaard Floer homology was introduced by Ozsváth and Szabó in 2000. Using Heegaard diagrams and symplectic geometry, this theory assigns a package of invariants

\[ \widehat{HF}(Y), HF^+(Y), HF^-(Y), HF^\infty(Y), \ldots \]

to a closed, oriented 3–manifold \( Y \).
Heegaard Floer homology was introduced by Ozsváth and Szabó in 2000. Using Heegaard diagrams and symplectic geometry, this theory assigns a package of invariants

\[ \widehat{HF}(Y), HF^+(Y), HF^-(Y), HF^\infty(Y), \ldots \]

to a closed, oriented 3–manifold \( Y \).

\( Y, \) closed 3-manifold \( \rightarrow (\Sigma, \alpha, \beta, z), \) Heegaard diagram
Heegaard Floer homology was introduced by Ozsváth and Szabó in 2000. Using Heegaard diagrams and symplectic geometry, this theory assigns a package of invariants

\[ \widehat{HF}(Y), HF^+(Y), HF^-(Y), HF^\infty(Y), \ldots \]

to a closed, oriented 3–manifold \( Y \).
Heegaard Floer homology was introduced by Ozsváth and Szabó in 2000. Using Heegaard diagrams and symplectic geometry, this theory assigns a package of invariants

$$\widehat{HF}(Y), \, HF^+(Y), \, HF^-(Y), \, HF^\infty(Y), \ldots$$

to a closed, oriented 3–manifold $Y$.

$Y$, closed 3-manifold $\rightarrow (\Sigma, \alpha, \beta, z)$, Heegaard diagram $\downarrow$

$HF(Y)$, homology $\leftarrow CF(Y)$, chain complex
Heegaard Floer homology was introduced by Ozsváth and Szabó in 2000. Using Heegaard diagrams and symplectic geometry, this theory assigns a package of invariants

$$\widehat{HF}(Y), HF^+(Y), HF^-(Y), HF^\infty(Y), \ldots$$

to a closed, oriented 3–manifold $Y$.

\[
\begin{array}{ccc}
Y, \text{ closed 3-manifold} & \rightarrow & (\Sigma, \alpha, \beta, z), \text{ Heegaard diagram} \\
\downarrow & & \downarrow \\
\widehat{HF}(Y), \text{ homology} & \leftarrow & CF(Y), \text{ chain complex}
\end{array}
\]
Other invariants

In the same spirit, one can construct a lot of invariants for low-dimensional objects.
Other invariants

In the same spirit, one can construct a lot of invariants for low-dimensional objects.

- knots and links: knot Floer homology $\widehat{HFK}(Y, K)$, (Ozsváth–Szabó, Rasmussen)
Other invariants

In the same spirit, one can construct a lot of invariants for low-dimensional objects.

- knots and links: knot Floer homology $\hat{HFK}(Y, K)$, (Ozsváth–Szabó, Rasmussen)
- 4-manifolds: mixed invariant, (Ozsváth–Szabó)
In the same spirit, one can construct a lot of invariants for low-dimensional objects.

- knots and links: knot Floer homology $\widehat{HFK}(Y, K)$, (Ozsváth–Szabó, Rasmussen)
- 4-manifolds: mixed invariant, (Ozsváth–Szabó)
- contact 3-manifolds: Ozsváth–Szabó contact invariant, (Ozsváth–Szabó, generalized by Etnyre and Honda–Kazez–Matić)
Other invariants

In the same spirit, one can construct a lot of invariants for low-dimensional objects.

- knots and links: knot Floer homology $\widehat{HF}(Y, K)$, (Ozsváth–Szabó, Rasmussen)
- 4-manifolds: mixed invariant, (Ozsváth–Szabó)
- contact 3-manifolds: Ozsváth–Szabó contact invariant, (Ozsváth–Szabó, generalized by Etnyre and Honda–Kazez–Matić)
- sutured 3-manifolds: sutured Floer homology, (Juhász)
In the same spirit, one can construct a lot of invariants for low-dimensional objects.

- knots and links: knot Floer homology $\widehat{HFK}(Y, K)$, (Ozsváth–Szabó, Rasmussen)
- 4-manifolds: mixed invariant, (Ozsváth–Szabó)
- contact 3-manifolds: Ozsváth–Szabó contact invariant, (Ozsváth–Szabó, generalized by Etnyre and Honda–Kazez–Matić)
- sutured 3-manifolds: sutured Floer homology, (Juhász)
- bordered 3-manifolds: bordered Floer homology, (Lipshitz–Ozsváth–Thurston)
Other invariants

In the same spirit, one can construct a lot of invariants for low-dimensional objects.

- knots and links: knot Floer homology $\widehat{\text{HFK}}(Y, K)$, (Ozsváth–Szabó, Rasmussen)
- 4-manifolds: mixed invariant, (Ozsváth–Szabó)
- contact 3-manifolds: Ozsváth–Szabó contact invariant, (Ozsváth–Szabó, generalized by Etnyre and Honda–Kazez–Matić)
- sutured 3-manifolds: sutured Floer homology, (Juhász)
- bordered 3-manifolds: bordered Floer homology, (Lipshitz–Ozsváth–Thurston)
- links in $S^3$: stable homotopy invariant, (Sarkar)
Other invariants

In the same spirit, one can construct a lot of invariants for low-dimensional objects.

- knots and links: knot Floer homology $\widehat{HFK}(Y, K)$, (Ozsváth–Szabó, Rasmussen)
- 4-manifolds: mixed invariant, (Ozsváth–Szabó)
- contact 3-manifolds: Ozsváth–Szabó contact invariant, (Ozsváth–Szabó, generalized by Etnyre and Honda–Kazez–Matić)
- sutured 3-manifolds: sutured Floer homology, (Juhász)
- bordered 3-manifolds: bordered Floer homology, (Lipshitz–Ozsváth–Thurston)
- links in $S^3$: stable homotopy invariant, (Sarkar)
- singular links, transverse and legendrian links...
Spin\(^c\) structures

The Spin\(^c\) structures on \(Y\) are in one-to-one correspondence with the elements in \(H^2(Y; \mathbb{Z})\). \(HF(Y)\) naturally splits with respect to Spin\(^c\) structures.

\[
HF(Y) \cong \bigoplus_{s \in \text{Spin}^c(Y)} HF(Y, s).
\]
Spin\(^c\) structures

The Spin\(^c\) structures on \(Y\) are in one-to-one correspondence with the elements in \(H^2(Y; \mathbb{Z})\). \(HF(Y)\) naturally splits with respect to Spin\(^c\) structures.

\[
HF(Y) \cong \bigoplus_{s \in \text{Spin}^c(Y)} HF(Y, s).
\]

Given \(h \in H_2(Y; \mathbb{Z})\), one can project the Spin\(^c\) structures to a line dual to \(h\). Let

\[
HF(Y, h, i) \cong \bigoplus_{s \in \text{Spin}^c(Y), \langle c_1(s), h \rangle = 2i} HF(Y, s),
\]
Spin\(^c\) structures

The Spin\(^c\) structures on \(Y\) are in one-to-one correspondence with the elements in \(H^2(Y; \mathbb{Z})\). \(HF(Y)\) naturally splits with respect to Spin\(^c\) structures.

\[
HF(Y) \cong \bigoplus_{s \in \text{Spin}^c(Y)} HF(Y, s).
\]

Given \(h \in H_2(Y; \mathbb{Z})\), one can project the Spin\(^c\) structures to a line dual to \(h\). Let

\[
HF(Y, h, i) \cong \bigoplus_{s \in \text{Spin}^c(Y), \langle c_1(s), h \rangle = 2i} HF(Y, s),
\]

then

\[
HF(Y) \cong \bigoplus_{i \in \mathbb{Z}} HF(Y, h, i).
\]
Spin$^c$ structures

The Spin$^c$ structures on $Y$ are in one-to-one correspondence with the elements in $H^2(Y;\mathbb{Z})$. $HF(Y)$ naturally splits with respect to Spin$^c$ structures.

$$HF(Y) \cong \bigoplus_{s \in Spin^c(Y)} HF(Y, s).$$

Given $h \in H_2(Y;\mathbb{Z})$, one can project the Spin$^c$ structures to a line dual to $h$. Let

$$HF(Y, h, i) \cong \bigoplus_{s \in Spin^c(Y), \langle c_1(s), h \rangle = 2i} HF(Y, s),$$

then

$$HF(Y) \cong \bigoplus_{i \in \mathbb{Z}} HF(Y, h, i).$$

Similarly, if $F$ is a Seifert surface for a knot $K \subset Y$, then

$$\hat{HFK}(Y, K) \cong \bigoplus_{i \in \mathbb{Z}} \hat{HFK}(Y, K, [F], i).$$
Thurston norm

Suppose $S$ is a compact surface with components $S_1, \ldots, S_n$, then its *norm*

$$x(S) = \sum_i \max\{-\chi(S_i), 0\}.$$
Suppose $S$ is a compact surface with components $S_1, \ldots, S_n$, then its norm

$$x(S) = \sum_i \max\{-\chi(S_i), 0\}.$$ 

Given $h \in H_2(Y)$, the Thurston norm of $h$ is defined to be

$$x(h) = \min\{x(S) \mid S \subset Y, \ S \text{ represents } h\}.$$
Heegaard Floer homology detects Thurston norm

It is known that Seiberg–Witten theory detects Thurston norm (Kronheimer–Mrowka). Ozsváth and Szabó proved an analogous result in Heegaard Floer homology.
Heegaard Floer homology detects Thurston norm

It is known that Seiberg–Witten theory detects Thurston norm (Kronheimer–Mrowka). Ozsváth and Szabó proved an analogous result in Heegaard Floer homology.

**Theorem (Ozsváth–Szabó)**

Suppose $Y$ is a closed 3–manifold, $h \in H_2(Y)$. Then

$$x(h) = 2 \max \{i \mid \hat{HF}(Y, h, i) \neq 0\}.$$
Heegaard Floer theory detects Thurston norm

It is known that Seiberg–Witten theory detects Thurston norm (Kronheimer–Mrowka). Ozsváth and Szabó proved an analogous result in Heegaard Floer homology.

Theorem (Ozsváth–Szabó)
Suppose $Y$ is a closed 3–manifold, $h \in H_2(Y)$. Then

$$x(h) = 2 \max \{i \mid \widehat{HF}(Y, h, i) \neq 0\}.$$ 

Theorem (Ozsváth–Szabó)
Suppose $K \subset Y$ is a null-homologous knot, $F$ is a minimal genus Seifert surface for $K$. Then

$$g(F) = \max \{i \mid \widehat{HFK}(Y, K, [F], i) \neq 0\}.$$
Fibered knots

Theorem (Ghiggini,Ni)

Suppose $K \subset Y$ is a null-homologous knot such that $Y - K$ is irreducible. Let $F$ be a Seifert surface for $K$. If

$$\widehat{HFK}(Y, K, [F], g(F)) \cong \mathbb{Z},$$

then $Y - K$ fibers over $S^1$ with fiber $F$. 

---

A 3-manifold is irreducible if any two-sphere in the manifold bounds a three-ball. In the original proof, contact and symplectic topology were used as Ozsváth and Szabó did in earlier works. This ingredient was replaced by Juhász by more elementary means.
Fibered knots

Theorem (Ghiggini,Ni)

Suppose $K \subset Y$ is a null-homologous knot such that $Y - K$ is irreducible. Let $F$ be a Seifert surface for $K$. If

$$\widehat{HF}(Y, K, [F], g(F)) \cong \mathbb{Z},$$

then $Y - K$ fibers over $S^1$ with fiber $F$.

A 3-manifold is irreducible if any two-sphere in the manifold bounds a three-ball.
Theorem (Ghiggini,Ni)

Suppose $K \subset Y$ is a null-homologous knot such that $Y - K$ is irreducible. Let $F$ be a Seifert surface for $K$. If

$$\widehat{HFK}(Y, K, [F], g(F)) \cong \mathbb{Z},$$

then $Y - K$ fibers over $S^1$ with fiber $F$.

A 3-manifold is irreducible if any two-sphere in the manifold bounds a three-ball.

In the original proof, contact and symplectic topology were used as Ozsváth and Szabó did in earlier works. This ingredient was replaced by Juhász by more elementary means.
Fibered 3–manifolds

Theorem (Ni $g > 1$, $A_i = 1$)

Suppose $Y$ is a closed irreducible 3–manifold, $F \subset Y$ is a closed connected surface. If $HF(Y, [F], g(F) - 1)$ is one–dimensional, then $Y$ is a surface bundle over the circle and $F$ is the fiber.

Some Applications of Heegaard Floer homology to Dehn surgery
Fibered 3–manifolds

**Theorem (Ni \( g > 1 \), Ai–Ni \( g = 1 \))**

Suppose \( Y \) is a closed irreducible 3–manifold, \( F \subset Y \) is a closed connected surface. If \( HF^+(Y, [F], g(F) - 1) \) is one–dimensional, then \( Y \) is a surface bundle over the circle and \( F \) is the fiber.
Ozsváth and Szabó defined twisted Heegaard Floer homology. Let us consider a special version of this theory, which is also called perturbed Heegaard Floer homology in the literature.
Ozsváth and Szabó defined twisted Heegaard Floer homology. Let us consider a special version of this theory, which is also called perturbed Heegaard Floer homology in the literature.

Let

\[ \Lambda = \left\{ \sum_{r \in \mathbb{R}} a_r T^r \middle| a_r \in \mathbb{R}, \# \{ a_r \neq 0, r \leq c \} < \infty \text{ for any } c \in \mathbb{R} \right\} \]

be the universal Novikov ring, which is actually a field.
In addition to the usual Heegaard diagram \((\Sigma, \alpha, \beta, z)\), we choose a 1-cycle \(\omega\) on \(\Sigma\).
Heegaard diagrams

\[(\Sigma, \alpha, \)\]
In addition to the usual Heegaard diagram \((\Sigma, \alpha, \beta, \gamma)\), we choose a 1-cycle \(\omega\) on \(\Sigma\).
Heegaard diagrams

$(\Sigma, \alpha, \beta, z), \quad \text{z}^*$
In addition to the usual Heegaard diagram 

\[(\Sigma, \alpha, \beta, z),\]

we choose a 1-cycle \(\omega\) on \(\Sigma\).
As usual in Heegaard Floer theory, there are two Lagrangian tori

\[ T_\alpha = \alpha_1 \times \cdots \times \alpha_g, \]
\[ T_\beta = \beta_1 \times \cdots \times \beta_g \]

in

\[ \text{Sym}^g(\Sigma) = \Sigma^{\times g} / S(g). \]
As usual in Heegaard Floer theory, there are two Lagrangian tori

\[ T_\alpha = \alpha_1 \times \cdots \times \alpha_g, \]
\[ T_\beta = \beta_1 \times \cdots \times \beta_g \]
in

\[ \text{Sym}^g(\Sigma) = \Sigma^{\times g} / S(g). \]

Let \( \widehat{CF}(Y, \omega; \Lambda) \) be the \( \Lambda \)-module freely generated by \( x \), where

\[ x \in T_\alpha \cap T_\beta. \]
The twisted Heegaard Floer chain complex, II

If $\phi$ is a topological Whitney disk connecting $x$ to $y$, let $\partial_\alpha \phi = (\partial \phi) \cap T_\alpha$. We define

$$A(\phi) = (\partial_\alpha \phi) \cdot \omega.$$
The twisted Heegaard Floer chain complex, II

If $\phi$ is a topological Whitney disk connecting $x$ to $y$, let
$\partial_\alpha \phi = (\partial \phi) \cap T_\alpha$. We define

$$A(\phi) = (\partial_\alpha \phi) \cdot \omega.$$ 

Let

$$\partial : \widehat{CF}(Y, \omega; \Lambda) \to \widehat{CF}(Y, \omega; \Lambda)$$

be the boundary map defined by

$$\partial x = \sum_y \sum_{\phi \in \pi_2(x, y) \atop \mu(\phi) = 1, n_z(\phi) = 0} \#(M(\phi)/\mathbb{R}) T^{A(\phi)} y.$$
The twisted Heegaard Floer chain complex, II

If $\phi$ is a topological Whitney disk connecting $x$ to $y$, let $\partial_\alpha \phi = (\partial \phi) \cap T_\alpha$. We define

$$A(\phi) = (\partial_\alpha \phi) \cdot \omega.$$ 

Let

$$\partial : \widehat{CF}(Y, \omega; \Lambda) \to \widehat{CF}(Y, \omega; \Lambda)$$

be the boundary map defined by

$$\partial x = \sum_y \sum_{\phi \in \pi_2(x,y)} \#(\mathcal{M}(\phi) / \mathbb{R}) T^{A(\phi)} y.$$ 

The homology of this chain complex is $\widehat{HF}(Y, \omega; \Lambda)$. It depends on $\omega$ only through its homology class in $Y$. 

Yi Ni

Some Applications of Heegaard Floer homology to Dehn surgery
Above is a genus 1 Heegaard diagram of $S^1 \times S^2$. There are two bigons $D_1, D_2$ connecting $\mathbf{x}$ to $\mathbf{y}$.

\[
A(D_1) = 1, \quad A(D_2) = 0.
\]
Above is a genus 1 Heegaard diagram of \( S^1 \times S^2 \). There are two bigons \( D_1, D_2 \) connecting \( x \) to \( y \).

\[
A(D_1) = 1, \quad A(D_2) = 0.
\]
Above is a genus 1 Heegaard diagram of $S^1 \times S^2$. There are two bigons $D_1, D_2$ connecting $x$ to $y$.

$$A(D_1) = 1, \quad A(D_2) = 0.$$
Non-separating two-spheres

If we work with untwisted coefficients, then the two bigons will cancel with each other in the boundary map:

$$\partial x = y - y = 0$$
Non-separating two-spheres

In twisted Floer homology, we have

$$\partial x = T^1 \cdot y - T^0 \cdot y$$
In twisted Floer homology, we have
\[ \partial x = T^1 \cdot y - T^0 \cdot y \]
\[ = (T - 1) \cdot y. \]
Non-separating two-spheres

In twisted Floer homology, we have

\[ \partial x = T^1 \cdot y - T^0 \cdot y \]

\[ = (T - 1) \cdot y. \]

Since \( T - 1 \) is invertible in \( \Lambda \), the two generators cancel. So

\[ \widehat{HF}(S^1 \times S^2, \omega; \Lambda) \cong 0. \]
Non-separating two-spheres

In twisted Floer homology, we have

$$\partial x = T^1 \cdot y - T^0 \cdot y$$

$$= (T - 1) \cdot y.$$ 

Since $T - 1$ is invertible in $\Lambda$, the two generators cancel. So

$$\widehat{HF}(S^1 \times S^2, \omega; \Lambda) \cong 0.$$ 

In general, if a manifold $Y$ contains a non-separating two-sphere $S$, and $\omega$ is 1-cycle in $Y$ such that $\omega \cdot S \neq 0$, then

$$\widehat{HF}(Y, \omega; \Lambda) \cong 0.$$
Heegaard Floer homology

Dehn surgery

Cosmetic surgeries

Property G
Definition: Dehn surgery

Given a knot $K$ in $Y$, we can remove a tubular neighborhood $N(K)$ of $K$ from $Y$, then glue in a solid torus $D^2 \times S^1$ by a homeomorphism

$$f: (\partial D^2) \times S^1 \to \partial(Y - N(K))$$

to get a new manifold. This process is called a Dehn surgery on the knot.
The homeomorphism type of the manifold obtained by Dehn surgery is determined by the homology class of $f(\partial D^2 \times \text{point})$.

Let $\mu$ be the meridian of the knot $K$, $\lambda$ be a longitude of $K$, then $f(\partial D^2 \times \text{point})$ is homologous to $q[\mu] + p[\lambda]$ for some $p, q \in \mathbb{Z}$. We call $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ the slope of the surgery. The new manifold is denoted by $Y_{p/q}(K)$. 
The homeomorphism type of the manifold obtained by Dehn surgery is determined by the homology class of

\[ f(\partial D^2 \times \text{point}). \]

Let \( \mu \) be the meridian of the knot \( K \), \( \lambda \) be a longitude of \( K \), then \( f(\partial D^2 \times \text{point}) \) is homologous to \( q[\mu] + p[\lambda] \) for some \( p, q \in \mathbb{Z} \). We call \( \frac{p}{q} \in \mathbb{Q} \cup \{\infty\} \) the slope of the surgery. The new manifold is denoted by \( Y_{p/q}(K) \).

The trivial surgery has slope \( 1/0 = \infty \).
The homeomorphism type of the manifold obtained by Dehn surgery is determined by the homology class of

\[ f(\partial D^2 \times \text{point}). \]

Let \( \mu \) be the meridian of the knot \( K \), \( \lambda \) be a longitude of \( K \), then \( f(\partial D^2 \times \text{point}) \) is homologous to \( q[\mu] + p[\lambda] \) for some \( p, q \in \mathbb{Z} \). We call \( \frac{p}{q} \in \mathbb{Q} \cup \{\infty\} \) the slope of the surgery. The new manifold is denoted by \( Y_{p/q}(K) \).

The trivial surgery has slope \( 1/0 = \infty \). When \( K \) is a null-homologous knot, there is a canonical longitude. The slope of this longitude is \( 0/1 = 0 \).
Surgery exact triangle in Heegaard Floer homology

The next theorem is a sample of the surgery exact sequences in Heegaard Floer homology.

\[ \text{Theorem (Ozsváth–Szabó)} \]

Suppose $K$ is a knot in $Y$, then there is an exact triangle relating the Heegaard Floer homology of $Y$, $Y_n(K)$, $Y_{n+1}(K)$ for any integer $n$:

\[ \text{HF}(Y) \rightarrow \text{HF}(Y_n(K)) \rightarrow \text{HF}(Y_{n+1}(K)). \]

Some Applications of Heegaard Floer homology to Dehn surgery
The next theorem is a sample of the surgery exact sequences in Heegaard Floer homology.

**Theorem (Ozsváth–Szabó)**

Suppose $K$ is a knot in $Y$, then there is an exact triangle relating the Heegaard Floer homology of $Y$, $Y_n(K)$, $Y_{n+1}(K)$ for any integer $n$:

$$
\begin{align*}
\text{HF}(Y) & \rightarrow \text{HF}(Y_n(K)) \\
& \rightarrow \text{HF}(Y_{n+1}(K)).
\end{align*}
$$

Some Applications of Heegaard Floer homology to Dehn surgery
The general surgery formula

Based on the surgery exact triangle, Ozsváth and Szabó proved a general surgery formula. Basically, if one knows the knot Floer chain complex associated to a pair \((Y, K)\), then one can compute the Heegaard Floer homology of all \(Y_{p/q}(K)\).
The general surgery formula

Based on the surgery exact triangle, Ozsváth and Szabó proved a general surgery formula. Basically, if one knows the knot Floer chain complex associated to a pair \((Y, K)\), then one can compute the Heegaard Floer homology of all \(Y_{p/q}(K)\).

**Theorem (Ozsváth–Szabó)**

Suppose \(K \subset Y\) is a knot in a homology sphere. There exist two chain complexes \(A(Y, K), B(Y, K)\), where \(A(Y, K)\) depends on the knot Floer chain complex, and \(B(Y, K)\) is a direct sum of infinitely many \(\text{CF}(Y)\). For any \(p, q\), \(\text{HF}(Y_{p/q}(K))\) is isomorphic to the homology of a mapping cone

\[
\bigoplus_{i=1}^{q} A(Y, K) \rightarrow \bigoplus_{j=1}^{p} B(Y, K).
\]
However, the algebra involved here is often too complicated.
However, the algebra involved here is often too complicated.

In practice people always require that the ambient manifold $Y$ has "simple" Floer homology, hence the homology of $B(Y, K)$ is simple.
L–spaces

A rational homology sphere $Y$ is called an $L$–space, if the rank of $\widehat{HF}(Y)$ is equal to $|H_1(Y;\mathbb{Z})|$. 
A rational homology sphere $Y$ is called an $L$–space, if the rank of $\hat{HF}(Y)$ is equal to $|H_1(Y; \mathbb{Z})|$.

Examples.
$S^3$, 
A rational homology sphere $Y$ is called an \textit{L–space}, if the rank of $\widehat{HF}(Y)$ is equal to $|H_1(Y; \mathbb{Z})|$. 

**Examples.**

$S^3$, $\widehat{HF}(S^3) \cong \mathbb{Z}$,
A rational homology sphere $Y$ is called an $L$–space, if the rank of $\widehat{HF}(Y)$ is equal to $|H_1(Y;\mathbb{Z})|$.

**Examples.**

$S^3$, $\widehat{HF}(S^3) \cong \mathbb{Z}$,

lens spaces $L(p,q)$,
A rational homology sphere $Y$ is called an $L$–space, if the rank of $\widehat{HF}(Y)$ is equal to $|H_1(Y;\mathbb{Z})|$. 

**Examples.**

$S^3$, $\widehat{HF}(S^3) \cong \mathbb{Z}$, lens spaces $L(p, q)$, $\widehat{HF}(L(p, q)) \cong \mathbb{Z}^p$. 

*Some Applications of Heegaard Floer homology to Dehn surgery*
A rational homology sphere $Y$ is called an $L$–space, if the rank of $\widehat{HF}(Y)$ is equal to $|H_1(Y; \mathbb{Z})|$.

**Examples.**

$S^3$, $\widehat{HF}(S^3) \cong \mathbb{Z}$,

lens spaces $L(p, q)$, $\widehat{HF}(L(p, q)) \cong \mathbb{Z}^p$,

spherical manifolds,
A rational homology sphere $Y$ is called an $L$–space, if the rank of $\widehat{HF}(Y)$ is equal to $|H_1(Y; \mathbb{Z})|$.

**Examples.**

$S^3$, $\widehat{HF}(S^3) \cong \mathbb{Z}$,
lens spaces $L(p, q)$, $\widehat{HF}(L(p, q)) \cong \mathbb{Z}^p$,
spherical manifolds,
double branched covers of $S^3$ over quasi-alternating links,
A rational homology sphere $Y$ is called an \textit{$L$–space}, if the rank of $\widehat{HF}(Y)$ is equal to $|H_1(Y; \mathbb{Z})|$. 

\textbf{Examples.}

$S^3$, $\widehat{HF}(S^3) \cong \mathbb{Z}$, 

lens spaces $L(p, q)$, $\widehat{HF}(L(p, q)) \cong \mathbb{Z}^p$, 

spherical manifolds, 

double branched covers of $S^3$ over quasi-alternating links, 

large surgeries on knots that admits $L$-space surgeries,
A rational homology sphere $Y$ is called an $L$–space, if the rank of $\widehat{HF}(Y)$ is equal to $|H_1(Y; \mathbb{Z})|$.

**Examples.**

- $S^3$, $\widehat{HF}(S^3) \cong \mathbb{Z}$,
- lens spaces $L(p, q)$, $\widehat{HF}(L(p, q)) \cong \mathbb{Z}^p$,
- spherical manifolds,
- double branched covers of $S^3$ over quasi-alternating links,
- large surgeries on knots that admits $L$-space surgeries,
- ...
A rational homology sphere $Y$ is called an \( L\)-space, if the rank of $\hat{\text{HF}}(Y)$ is equal to $|H_1(Y; \mathbb{Z})|$.

**Examples.**

$S^3$, $\hat{\text{HF}}(S^3) \cong \mathbb{Z}$,

lens spaces $L(p, q)$, $\hat{\text{HF}}(L(p, q)) \cong \mathbb{Z}^p$,

spherical manifolds,

double branched covers of $S^3$ over quasi-alternating links,

large surgeries on knots that admits $L$-space surgeries,

\ldots

For a knot in an $L$-space, the surgery formula is much easier.
If a manifold contains a non-separating two-sphere $S$, then the twisted Heegaard Floer homology is 0. Thus it is easier to study the surgery formula for knots in such manifolds.
If a manifold contains a non-separating two-sphere $S$, then the twisted Heegaard Floer homology is 0. Thus it is easier to study the surgery formula for knots in such manifolds. Roughly speaking, for $p, q > 0$,

$$HF(Y_{p/q}(K), \omega; \Lambda) = H_*(qA(Y, K, \omega; \Lambda) \to pB(Y, K, \omega; \Lambda))$$
Non-separating spheres

If a manifold contains a non-separating two-sphere $S$, then the twisted Heegaard Floer homology is 0. Thus it is easier to study the surgery formula for knots in such manifolds. Roughly speaking, for $p, q > 0$,

$$HF(Y_{p/q}(K), \omega; \Lambda) \equiv H_*(qA(Y, K, \omega; \Lambda) \rightarrow pB(Y, K, \omega; \Lambda)) = H_*(qA(Y, K, \omega; \Lambda))$$
Non-separating spheres

If a manifold contains a non-separating two-sphere $S$, then the twisted Heegaard Floer homology is 0. Thus it is easier to study the surgery formula for knots in such manifolds. Roughly speaking, for $p, q > 0$,

$$HF\left(\frac{Y_p}{q}(K), \omega; \Lambda\right) = H_*(qA(Y, K, \omega; \Lambda) \rightarrow pB(Y, K, \omega; \Lambda)) = H_*(qA(Y, K, \omega; \Lambda)) = qH_*(A(Y, K, \omega; \Lambda))$$
- Heegaard Floer homology
- Dehn surgery
- Cosmetic surgeries
- Property G
Cosmetic surgeries

Definition

Suppose \( K \subset Y \) is a knot, \( r, s \) are two slopes on \( K \). Call two surgeries with slopes \( r, s \) **cosmetic** if there is a homeomorphism between \( Y_r(K) \) and \( Y_s(K) \). If the homeomorphism is orientation preserving, then the two surgeries are **purely cosmetic**.

Theorem (Gordon–Luecke)

Suppose \( L \subset S^3 \) is a non-trivial knot. Then \( S^3 \) is not homeomorphic to \( S^3 \) for any \( r \in \mathbb{Q} \). In other words, \( S^3 \) can not be obtained via cosmetic surgeries on any knot \( K \subset Y \) unless the complement of \( K \) is a solid torus.
Cosmetic surgeries

Definition
Suppose $K \subset Y$ is a knot, $r, s$ are two slopes on $K$. Call two surgeries with slopes $r, s$ cosmetic if there is a homeomorphism between $Y_r(K)$ and $Y_s(K)$. If the homeomorphism is orientation preserving, then the two surgeries are purely cosmetic.

Theorem (Gordon–Luecke)

Suppose $L \subset S^3$ is a non-trivial knot. Then $S^3_r(L)$ is not homeomorphic to $S^3$ for any $r \in \mathbb{Q}$. 
Cosmetic surgeries

Definition

Suppose $K \subset Y$ is a knot, $r, s$ are two slopes on $K$. Call two surgeries with slopes $r, s$ cosmetic if there is a homeomorphism between $Y_r(K)$ and $Y_s(K)$. If the homeomorphism is orientation preserving, then the two surgeries are purely cosmetic.

Theorem (Gordon–Luecke)

Suppose $L \subset S^3$ is a non-trivial knot. Then $S^3_r(L)$ is not homeomorphic to $S^3$ for any $r \in \mathbb{Q}$. In other words, $S^3$ can not be obtained via cosmetic surgeries on any knot $K \subset Y$ unless the complement of $K$ is a solid torus.
Examples

\[ S_3(r(K)) \sim S_3 - r(K) \text{ for any } r \in \mathbb{Q} \text{ and any amphichiral knot } K. \]

Mathieu: \[ S_3(18k + 9)/ (3k + 1)(T) \sim S_3(18k + 9)/ (3k + 2)(T) \] for any nonnegative integer \( k \), where \( T \) is the right-hand trefoil.
Examples

\[ S_r^3(K) \cong -S_{-r}^3(K) \text{ for any } r \in \mathbb{Q} \text{ and any amphichiral knot } K. \]
Examples

\[ S^3_r(K) \cong -S^3_{-r}(K) \] for any \( r \in \mathbb{Q} \) and any amphichiral knot \( K \).

Mathieu: \( S^3_{(18k+9)/(3k+1)}(T) \cong -S^3_{(18k+9)/(3k+2)}(T) \) for any nonnegative integer \( k \), where \( T \) is the right-hand trefoil.
Conjecture (Bleiler, Kirby’s List, Problem 1.81)

Suppose $K$ is a knot in a closed manifold $Y$.
(1) If two surgeries are purely cosmetic, then there is a homeomorphism of $Y - K$ which takes one slope to the other.
(2) If the complement of $K$ is not the solid torus, then there are no purely cosmetic surgeries on $K$. 

Knots in $S^3$

**Theorem (Ozsváth–Szabó)**

Suppose $K \subset S^3$ is a knot. If two positive rational numbers $r, s$ satisfy that $S^3_r(K) \sim \pm S^3_s(K)$, then either $r = s$ or $S^3_r(K)$ is an L-space.

**Theorem (Ghiggini)**

Suppose $K \subset S^3$ is a genus-1 knot. If $K$ admits an L-space surgery, then $K$ is the trefoil knot.

Yi Ni

Some Applications of Heegaard Floer homology to Dehn surgery
Theorem (Ozsváth–Szabó)

Suppose $K \subset S^3$ is a knot. If two positive rational numbers $r, s$ satisfy that $S^3_r(K) \cong \pm S^3_s(K)$, then either $r = s$ or $S^3_r(K)$ is an L-space.
Knots in $S^3$

Theorem (Ozsváth–Szabó)

Suppose $K \subset S^3$ is a knot. If two positive rational numbers $r, s$ satisfy that $S^3_1(K) \cong \pm S^3_s(K)$, then either $r = s$ or $S^3_r(K)$ is an L-space.

Theorem (Ghiggini)

Suppose $K \subset S^3$ is a genus-1 knot. If $K$ admits an L-space surgery, then $K$ is the trefoil knot.
The rank of $\widehat{HF}(S^3_{p/q}(K))$

$$H_1(S^3_{p/q}(K); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z},$$ so it determines the number $p$. 

Using the surgery formula, one can get a formula for the rank of $\widehat{HF}(S^3_{p/q}(K))$. Fix $p > 0$, this rank is non-decreasing as $q$ increases.

Moreover, $\text{rank} \; \widehat{HF}(S^3_{p/q_1}(K)) = \text{rank} \; \widehat{HF}(S^3_{p/q_2}(K))$ for $0 < q_1 < q_2$ only if $K$ admits an $L$-space surgery.
The rank of $\hat{HF}(S^3_{p/q}(K))$

$H_1(S^3_{p/q}(K); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, so it determines the number $p$.

Using the surgery formula, one can get a formula for the rank of $\hat{HF}(S^3_{p/q}(K))$. 
The rank of $\widehat{HF}(S^3_{p/q}(K))$

\[ H_1(S^3_{p/q}(K); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}, \] so it determines the number $p$.

Using the surgery formula, one can get a formula for the rank of $\widehat{HF}(S^3_{p/q}(K))$. Fix $p > 0$, this rank is non-decreasing as $q > 0$ increases.
The rank of $\hat{HF}(S^3_{p/q}(K))$

$H_1(S^3_{p/q}(K); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, so it determines the number $p$.

Using the surgery formula, one can get a formula for the rank of $\hat{HF}(S^3_{p/q}(K))$. Fix $p > 0$, this rank is non-decreasing as $q > 0$ increases. Moreover,

$$\text{rank } \hat{HF}(S^3_{p/q_1}(K)) = \text{rank } \hat{HF}(S^3_{p/q_2}(K))$$

for $0 < q_1 < q_2$ only if $K$ admits an $L$-space surgery.
Theorem (Ni)

Suppose $Y$ is a closed 3–manifold with $b_1(Y) > 0$, $K$ is a null-homologous knot in $Y$. Suppose $(Y, K)$ satisfies one of the following conditions.

1. $Y$ contains a non-separating two-sphere, and $Y - K$ is irreducible;
2. for any nonzero element $h \in H^2(Y) \subset H^2(Y - K)$, $\chi(Y)(h) < \chi(Y - K)(h)$;
3. $\chi(Y) \equiv 0$, while the restriction of $\chi(Y)$ on $H^2(Y)$ is nonzero.

The conclusion is, if two rational numbers $r, s$ satisfy that $Y_r(K) \sim = \pm Y_s(K)$, then $r = \pm s$. 
Theorem (Ni)

Suppose $Y$ is a closed 3–manifold with $b_1(Y) > 0$, $K$ is a null-homologous knot in $Y$. Suppose $(Y, K)$ satisfies one of the following conditions.

1. $Y$ contains a non-separating two-sphere, and $Y - K$ is irreducible;

Suppose two rational numbers $r, s$ satisfy that $Y_r(K) \sim = \pm Y_s(K)$, then $r = \pm s$. 

Some Applications of Heegaard Floer homology to Dehn surgery
Theorem (Ni)

Suppose $Y$ is a closed 3–manifold with $b_1(Y) > 0$, $K$ is a null-homologous knot in $Y$. Suppose $(Y, K)$ satisfies one of the following conditions.

(1) $Y$ contains a non-separating two-sphere, and $Y - K$ is irreducible;

(2) for any nonzero element $h \in H_2(Y) \subset H_2(Y - K)$,

$$x_Y(h) < x_{Y-K}(h);$$
Theorem (Ni)

Suppose $Y$ is a closed 3–manifold with $b_1(Y) > 0$, $K$ is a null-homologous knot in $Y$. Suppose $(Y, K)$ satisfies one of the following conditions.

1. $Y$ contains a non-separating two-sphere, and $Y - K$ is irreducible;
2. for any nonzero element $h \in H_2(Y) \subset H_2(Y - K)$,
   
   $x_Y(h) < x_{Y-K}(h)$;
3. $x_Y \equiv 0$, while the restriction of $x_{Y-K}$ on $H_2(Y)$ is nonzero.

Some Applications of Heegaard Floer homology to Dehn surgery
Theorem (Ni)

Suppose \( Y \) is a closed 3–manifold with \( b_1(Y) > 0 \), \( K \) is a null-homologous knot in \( Y \). Suppose \((Y, K)\) satisfies one of the following conditions.

1. \( Y \) contains a non-separating two-sphere, and \( Y - K \) is irreducible;
2. for any nonzero element \( h \in H_2(Y) \subset H_2(Y - K) \),
   \[ x_Y(h) < x_{Y-K}(h); \]
3. \( x_Y \equiv 0 \), while the restriction of \( x_{Y-K} \) on \( H_2(Y) \) is nonzero.

The conclusion is, if two rational numbers \( r, s \) satisfy that \( Y_r(K) \cong \pm Y_s(K) \), then \( r = \pm s \).
Idea of the proof of (1)

Suppose $r = \frac{p}{q}, \ q > 0$. 

Since $K$ is null-homologous, $|p|$ is determined by $H_1(Y_{\frac{p}{q}}(K))$. From previous surgery formula $\text{rank} \hat{HF}(Y_{\frac{p}{q}}(K), \omega; \Lambda) = q \cdot \text{rank} H^*(A(Y, K), \omega; \Lambda)$, so $q$ is determined by $\text{rank} \hat{HF}(Y_{\frac{p}{q}}(K), \omega; \Lambda)$ provided it is nonzero.
Idea of the proof of (1)

Suppose $r = \frac{p}{q}$, $q > 0$.
Since $K$ is null-homologous, $|p|$ is determined by $H_1(Y_{p/q}(K))$. 
Idea of the proof of (1)

Suppose $r = \frac{p}{q}$, $q > 0$. Since $K$ is null-homologous, $|p|$ is determined by $H_1(Y_{p/q}(K))$.

From previous surgery formula

$$\text{rank } \widehat{HF}(Y_{p/q}(K), \omega; \Lambda) = q \cdot \text{rank } H_\ast(A(Y, K, \omega; \Lambda)),$$

so $q$ is determined by $\text{rank } \widehat{HF}(Y_{p/q}(K), \omega; \Lambda)$ provided it is nonzero.
- Heegaard Floer homology
- Dehn surgery
- Cosmetic surgeries
- Property G
Property R

The following “Property R” was conjectured by Poénaru and proved by Gabai.

**Theorem (Gabai)**

*Suppose $K$ is a knot in $S^3$. If the zero surgery on $K$ yields $S^1 \times S^2$, then $K$ is the unknot.*
The following “Property R” was conjectured by Poénaru and proved by Gabai.

**Theorem (Gabai)**

Suppose $K$ is a knot in $S^3$. If the zero surgery on $K$ yields $S^1 \times S^2$, then $K$ is the unknot.

**Theorem (Gabai)**

Suppose $K$ is a knot in $S^3$, $F$ is a minimal genus Seifert surface for $K$. 
The following “Property R” was conjectured by Poénaru and proved by Gabai.

**Theorem (Gabai)**

*Suppose $K$ is a knot in $S^3$. If the zero surgery on $K$ yields $S^1 \times S^2$, then $K$ is the unknot.*

**Theorem (Gabai)**

*Suppose $K$ is a knot in $S^3$, $F$ is a minimal genus Seifert surface for $K$. Let $\hat{F} \subset S^3_0(K)$ be the surface obtained by capping off $\partial F$ with a disk, then $\hat{F}$ is Thurston norm minimizing in $S^3_0(K)$.*
The following “Property R” was conjectured by Poénaru and proved by Gabai.

**Theorem (Gabai)**

Suppose $K$ is a knot in $S^3$. If the zero surgery on $K$ yields $S^1 \times S^2$, then $K$ is the unknot.

**Theorem (Gabai)**

Suppose $K$ is a knot in $S^3$, $F$ is a minimal genus Seifert surface for $K$. Let $\hat{F} \subset S^3_0(K)$ be the surface obtained by capping off $\partial F$ with a disk, then $\hat{F}$ is Thurston norm minimizing in $S^3_0(K)$. Moreover, if $S^3_0(K)$ fibers over the circle, then $K$ is a fibered knot.
Motivated by the above theorem, we define the following "Property G".

**Definition**

Suppose $K$ is a null-homologous knot in a closed 3–manifold $Y$. We say $K$ has **Property G**, if the following conditions hold:

1. (G1) any minimal genus Seifert surface for $K$ extends to a Thurston norm minimizing surface in $Y$ after attaching a disk to its boundary;
2. (G2) if $Y_0(K)$ fibers over $S^1$, such that the homology class of the fiber is the extension of the homology class of a Seifert surface $F$ for $K$, then $K$ is a fibered knot, and the homology class of the fiber is $[F]$. 

Yi Ni
Motivated by the above theorem, we define the following “Property G”.

Definition
Suppose $K$ is a null-homologous knot in a closed 3–manifold $Y$. We say $K$ has Property G, if the following conditions hold:

(G1) any minimal genus Seifert surface for $K$ extends to a Thurston norm minimizing surface in $Y_0(K)$ after attaching a disk to its boundary;
Motivated by the above theorem, we define the following “Property G”.

**Definition**

Suppose $K$ is a null-homologous knot in a closed 3–manifold $Y$. We say $K$ has **Property G**, if the following conditions hold:

1. **(G1)** any minimal genus Seifert surface for $K$ extends to a Thurston norm minimizing surface in $Y_0(K)$ after attaching a disk to its boundary;
2. **(G2)** if $Y_0(K)$ fibers over $S^1$, such that the homology class of the fiber is the extension of the homology class of a Seifert surface $F$ for $K$, then $K$ is a fibered knot, and the homology class of the fiber is $[F]$. 
Using an argument of Ozsváth and Szabó, one can prove the following result.

Proposition
Suppose $Y$ is an $L$–space, $K \subset Y$ is a null-homologous knot. If $g(K) > 1$, then $\hat{HFK}(Y, K, g) \cong HF^+(Y_0(K), g-1)$.

There is a similar statement when $g(K) = 1$ with twisted coefficients.

Corollary (Ni, Ai–Ni)
Null-homologous knots in $L$-spaces have Property G.
Knots in $L$-spaces

Using an argument of Ozsváth and Szabó, one can prove the following result.

**Proposition**

*Suppose $Y$ is an $L$–space, $K \subset Y$ is a null-homologous knot. If $g(K) > 1$, then*

\[ \widehat{HFK}(Y, K, g) \cong HF^+(Y_0(K), g - 1). \]
Knots in $L$-spaces

Using an argument of Ozsváth and Szabó, one can prove the following result.

**Proposition**

*Suppose $Y$ is an $L$–space, $K \subset Y$ is a null-homologous knot. If $g(K) > 1$, then*

\[
\widehat{\text{HFK}}(Y, K, g) \cong \text{HF}^+(Y_0(K), g - 1).
\]

There is a similar statement when $g(K) = 1$ with twisted coefficients.
Knots in $L$-spaces

Using an argument of Ozsváth and Szabó, one can prove the following result.

**Proposition**

Suppose $Y$ is an $L$–space, $K \subset Y$ is a null-homologous knot. If $g(K) > 1$, then

$$\widehat{HFK}(Y, K, g) \cong HF^+(Y_0(K), g - 1).$$

There is a similar statement when $g(K) = 1$ with twisted coefficients.

**Corollary (Ni, Ai–Ni)**

Null-homologous knots in $L$-spaces have Property G.
Knots in reducible manifolds

Theorem (Gabai)
Suppose $K$ is a null-homologous knot in a reducible manifold $Y$ such that $Y - K$ is irreducible. Suppose further that $H_1(Y;\mathbb{Z})$ is torsion-free. Then $K$ has Property G.

Theorem (Ni)
Suppose $K$ is a null-homologous knot in a manifold $Y$. Suppose $Y$ contains a non-separating sphere, and $Y - K$ is irreducible. Then $K$ has Property G.
Theorem (Gabai)

Suppose $K$ is a null-homologous knot in a reducible manifold $Y$ such that $Y - K$ is irreducible. Suppose further that $H_1(Y; \mathbb{Z})$ is torsion-free. Then $K$ has Property G.
Theorem (Gabai)

Suppose $K$ is a null-homologous knot in a reducible manifold $Y$ such that $Y - K$ is irreducible. Suppose further that $H_1(Y; \mathbb{Z})$ is torsion-free. Then $K$ has Property G.

Theorem (Ni)

Suppose $K$ is a null-homologous knot in a manifold $Y$. Suppose $Y$ contains a non-separating sphere, and $Y - K$ is irreducible. Then $K$ has Property G.
Theorem (Gabai)
Suppose $K$ is a null-homologous knot in a reducible manifold $Y$ such that $Y - K$ is irreducible. Suppose further that $H_1(Y; \mathbb{Z})$ is torsion-free. Then $K$ has Property G.

Theorem (Ni)
Suppose $K$ is a null-homologous knot in a manifold $Y$. Suppose $Y$ contains a non-separating sphere, and $Y - K$ is irreducible. Then $K$ has Property G.
Null-homotopic knots?

Conjecture (Boileau–Gabai)

Suppose $K$ is a null-homotopic knot in a closed manifold $Y$, then $K$ has Property G.
Thank you!