1. The notion of a topological space

Part of the rigorization of analysis in the 19th century was the realization that notions like convergence of sequences and continuity of functions (e.g. \( f : \mathbb{R}^n \to \mathbb{R}^m \)) were most naturally formulated by paying close attention to the mapping properties between subsets \( U \) of the domain and codomain with the property that when \( x \in U \), there exists \( \epsilon > 0 \) such that \( ||y - x|| < \epsilon \) implies \( y \in U \). Such sets are called open. In the early twentieth century it was realized that many of the constructions formerly regarded as “analytic” in nature could be carried out in a very general context of sets and maps between them, provided only that the sets are endowed with a distinguished family of subsets, decreed to be open, and satisfying some very mild axioms. This led to the notion of an abstract topological space, as follows.

Definition: A topological space is a pair \((X, \tau)\), where \( X \) is a set and \( \tau = \{U_i\}_{i \in I} \) is a family of subsets of \( X \) satisfying the following axioms:

(TS1) \( \emptyset, X \in \tau \).
(TS2) \( U_1, U_2 \in \tau \implies U_1 \cap U_2 \in \tau \).
(TS3) For any subset \( J \subset I \), \( \bigcup_{i \in J} U_i \in \tau \).

Moreover, a family \( \tau \subset 2^X \) of subsets of a set \( X \) satisfying (TS1)-(TS3) is called a topology on \( X \), and the elements of \( \tau \) are called open sets.

Axiom (TS2) evidently implies that the intersection of any finite number \( U_1, \ldots, U_n \) of elements of \( \tau \) remains an element of \( \tau \): we say \( \tau \) is closed under finite intersections. Similarly, we express (TS3) as saying that \( \tau \) is closed under arbitrary unions.

If \( (X, \tau_X) \) and \( (Y, \tau_Y) \) are topological spaces, a map \( f : X \to Y \) is continuous if for all \( V \in \tau_Y \), \( f^{-1}(V) \in \tau_X \). A function \( f : X \to Y \) between topological spaces is a homeomorphism if it is bijective, continuous, and has a continuous inverse. A function \( f \) is open if for \( U \in \tau_X \), \( f(U) \in \tau_Y \).

Exercise 1.1: For a function \( f : X \to Y \) between topological spaces \( (X, \tau_X) \) and \( (Y, \tau_Y) \), TFAE:

a) \( f \) is a homeomorphism.

b) \( f \) is bijective and for all \( V \subset Y \), \( V \) is open iff \( f^{-1}(V) \) is open in \( X \).

c) \( f \) is bijective and for all \( U \subset X \), \( U \) is open iff \( f(U) \) is open in \( X \).

d) \( f \) is bijective, continuous and open.
Tournant dangereuse: A continuous bijection is not in general a homeomorphism, although there are certain circumstances (other than openness of \( f \)) under which this becomes the case.

Exercise 1.2: Let \((X, \tau_X), (Y, \tau_Y), (Z, \tau_Z)\) be topological spaces and \(f : X \to Y\) and \(g : Y \to Z\) be continuous functions. Show that \(g \circ f : X \to Z\) is a continuous function from \((X, \tau_X)\) to \((Z, \tau_Z)\).

Thus we get a category by taking as objects the topological spaces and as morphisms the continuous maps between them. Note that a homeomorphism is precisely an isomorphism in the categorical sense.

Exercise 1.3: Let \((X, d)\) be a metric space. Show that the open subsets form a topology.

The topology \(\tau_d\) of Exercise 1.1 is called the metric topology. We also say that \(\tau_d\) is the topology induced by \(d\).

A topological space \((X, \tau)\) is metrizable if \(\tau = \tau_d\) is the metric topology for some metric \(d\). The use of a metric greatly simplifies many purely topological considerations, so it is of great interest to know necessary and sufficient conditions for metrizability. Indeed this was one of the main problems of General Topology. It is also a success story in that for some time now we have had rather satisfactory “metrization theorems” sufficient to show that most spaces of “classical interest” (e.g. manifolds, function spaces of various kinds) are metrizable. We should therefore mention at the outset that for various reasons it is crucial to consider the larger class of topological spaces. Here are some brief hints at why this is the case:

(i) One of the great themes of twentieth century mathematics has been the “geometrization” of algebraic objects of various kinds, which begins with the association of a topological space to an algebraic object. To name three basic (and mutually related) examples:

(a) The Gelfand-Naimark spectrum of maximal ideals of a commutative \(C^*\) algebra.
(b) The Stone space of ultrafilters on a Boolean algebra.
(c) The Zariski space of prime ideals of a commutative ring.

In cases (a) and (b) the spaces are often metrizable: the metrizability is equivalent to a certain countability condition (which can be phrased in terms of the original algebra and also in terms of the associated topological space). However, it is simpler, more general, and more natural to think in terms of arbitrary topological spaces. In the third case the associated topological spaces will never be metrizable (not even Hausdorff, to be defined shortly) unless the associated geometric object is “zero-dimensional.” But in all three cases, even though the associated space may be metrizable, the construction does not single out any particular metric.

(ii) A basic construction in algebra is the completion of a group \(G\) (or a ring...) with respect to a family of normal subgroups \(G_\alpha\) (ideals...). In this situation the Hausdorff axiom (a crude necessary condition for metrizability) is equivalent to the
condition $\cap_{\alpha} G_{\alpha} = \{1\}$. This condition is desirable but is not always satisfied—indeed, in many situations whether or not this condition holds is a deep and interesting question. As an example, when we take the $G_{\alpha}$’s to be the finite-index normal subgroups, this condition on $G$ is called residual finiteness and is a highly studied property in combinatorial group theory. (Even if this intersection condition holds, the completion need not be metrizable.)

(iii) It turns out to be very natural and important to consider arbitrary products of topological spaces, e.g. in the study of spaces of functions. Given any indexed family $(X_i, \tau_i)_{i \in I}$ of topological spaces, there is a canonical topology on the Cartesian product $X = \prod_{i \in I} X_i$. If each factor space $X_i$ is metrizable and the index set $I$ is countable, then the product topology on $X$ is metrizable. However, when the index set $X_i$ is uncountable—and the product is nondegenerate in the sense that each $X_i$ has more than one point—then the product topology is not metrizable.

Let us now give an example of a non-metrizable space, beginning with the following all-important definition.

Definition: A topological space $X$ is Hausdorff if given distinct points $x, y$ in $X$, there exist open sets $U \ni x, V \ni y$ such that $U \cap Y = \emptyset$.

Exercise 1.4: Show that the topology induced by any metric is Hausdorff.

Example 1.5: For a nonempty set $X$, $\tau = \{\emptyset, X\}$ is a topology on $X$, called the indiscrete, or trivial, topology. If $X$ has more than one element, this topology is evidently not Hausdorff.

Example 1.6: At the other extreme, for a nonempty set $X$, $\tau = 2^X$, the collection of all subsets of $X$, forms a topology, called the discrete topology.

Remark: The discrete and indiscrete topologies coincide iff $X$ has at most one element. Otherwise they are distinct and indeed give rise to non-homeomorphic spaces.

Exercise 1.7: Show that a topological space is discrete iff for all $x \in X$, $\{x\}$ is open.

A discrete topological space is metrizable: on any set $X$ consider the discrete metric $d(x, y) = 1 - \delta_{x, y}$, i.e., is 1 if $x \neq 1$ and is 0 if $x = y$. With this metric $B(x, \frac{1}{2}) = \{x\}$, so the discrete metric induces the discrete topology.

Exercise 1.8: Suppose $X$ is a finite topological space (i.e., $X$ is a finite set, which then forces $\tau \subset 2^X$ to be finite as well). If $X$ is Hausdorff, then it is discrete. In particular, finite metric spaces are discrete.

\footnote{The “canonicity” can be justified in terms of a universal mapping property: it is the product in the categorical sense, although it seems unlikely that the reader will have the category-theoretic background to appreciate this but not have seen the definition of the product topology! Anyway, there is a simple-to-define and useful topology on the Cartesian product.}
Example 1.9 (DVR topology): Consider the two element set $X = \{\circ, \bullet\}$. We take $\tau = \{\emptyset, \circ, X\}$. This gives a topology on $X$ in which the point $\circ$ is open but the point $\bullet$ is not, so $X$ is finite and nondiscrete, hence nonmetrizable.

Exercise 1.10: Let $X$ a set.

a) Show that, up to homeomorphism, there are precisely 3 topologies on a two-element set: the trivial topology, the discrete topology, and the DVR topology.

b) For $n \in \mathbb{Z}^+$, let $I(n)$ denote the number of isomorphism classes of topologies on $\{1, \ldots, n\}$. Show that $\lim_{n \to \infty} I(n) = \infty$. (Note that only one of these topologies, the discrete topology, is metrizable.)

d) Can you describe the asymptotics of $I(n)$, or even give reasonable lower and/or upper bounds?

Example 1.11 (cofinite topology): Let $X$ be an infinite set, and let $\tau$ consist of $\emptyset$ together with subsets whose complement is finite (or, for short, “cofinite subsets”). This is easily seen to form a topology, in which any two nonempty open sets intersect, hence a non-Hausdorff topology.

Example 1.12 (Moore plane): Let $X$ be the subset of $\mathbb{R}^2$ consisting of pairs $(x, y)$ with $y \geq 0$, endowed with the following “exotic” topology: a subset $U$ of $X$ is open if: whenever it contains a point $P = (x, y)$ with $y > 0$ it contains some open Euclidean disk $B(P, \epsilon)$; and whenever it contains a point $P = (x, 0)$ it contains $P \cup B((x, 0), \epsilon)$ for some $\epsilon > 0$, i.e., an open disk in the upper-half plane tangent to the $x$-axis at $P$. The Moore plane satisfies several properties shared by all metrizable spaces – it is first countable and Tychonoff – but not the property of normality. We will establish these facts later on when we possess the requisite vocabulary.

Exercise 1.13: Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces and $f : X \to Y$ be a map between them. Show that $f$ is continuous iff for all closed $A \subset Y$, $f^{-1}(A)$ is closed in $X$.

Exercise 1.14 (Zariski topology): Let $R$ be a commutative ring, and let $\text{Spec} R$ be the set of prime ideals of $R$. For any subset $S$ of $R$ (including $\emptyset$, let $C(S)$ be the set of prime ideals containing $S$.

a) Show that $C(S_1) \cup C(S_2) = C(S_1 \cap S_2)$.

b) Show that, for any collection $\{S_i\}_{i \in I}$ of subsets of $R$, $\bigcap_i C(S_i) = C(\bigcup_i S_i)$.

c) Note that $C(\emptyset) = \text{Spec} R$, $C(R) = \emptyset$.

Thus the $C(S)$’s form the closed sets for a topology, called the Zariski topology on $\text{Spec} R$.

d) If $\varphi : R \to R'$ is a homomorphism of commutative rings, show that $\varphi^* : \text{Spec} R' \to \text{Spec} R$, $P \mapsto \varphi^{-1}(P)$ is a continuous map.

e) Let $\text{rad}(R)$ be the radical of $R$. Show that the natural map $\text{Spec}(R/\text{rad}(R)) \to \text{Spec}(R)$ is a homeomorphism.

f) Let $R$ be a discrete valuation ring. Show that $\text{Spec} R$ is the topological space of

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2This question has received a lot of attention but is, to the best of my knowledge, open in general.

3When we say that two subsets intersect, we mean of course that their intersection is nonempty.
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Example X.X above.
g) Let $k$ be an algebraically closed field and $R = k[t]$. Show that Spec$(R)$ can, as a topological space, be identified with $k$ itself with the cofinite topology.

Exercise 1.15: For a topological space $X$, TFAE:
(a) Given any ordered pair $(x, y)$ of distinct points of $X$, there exists an open set $U$ containing $x$ but not $y$.
(b) For all points $x$ in $X$, $\{x\}$ is a closed subset.

It is traditional to call a space satisfying these equivalent conditions (T1). For various reasons we shall in these notes instead call this property separated. Separatedness and Hausdorffness are both instances of “separation axioms.” Later we will justify this terminology and consider similar but stronger axioms, which play a key role in deciding whether a space is metrizable.

Exercise 1.16: Say that a point $\eta$ in a topological space $X$ is a generic point if the only closed subset containing $\eta$ is $X$ itself.
a) Show that $X$ has a generic point iff it is not the union of two proper closed subsets; this property is called irreducibility.
b) Show that an irreducible (T1) space consists of a single point.
c) If $R$ is a commutative ring, show that Spec$R$ is irreducible iff $R/\text{rad}(R)$ is an integral domain.

2. Alternative characterizations of topological spaces

2.1. Closed sets. In a topological space $(X, \tau)$, define a closed subset to be a subset whose complement is open. Evidently specifying the open subsets is equivalent to specifying the closed subsets.

The closed subsets of a topological space satisfy the following properties:

(CTS1) $\emptyset$, $X$ are closed.
(CTS2) Finite unions of closed sets are closed.
(CTS3) Arbitrary intersections of closed sets are closed.

Conversely, given such a family of subsets of $X$, then taking the open sets as the complements of each element in this family, we get a topology.

2.2. Closure. If $S$ is a subset of a topological space, we define its closure $\overline{S}$ to be the intersection of all closed subsets containing $S$. Since $X$ itself is closed containing $S$, this intersection is nonempty, and a moment’s thought reveals it to be the minimal closed subset containing $S$.

Viewing closure as a mapping $c$ from $2^X$ to itself, it satisfies the following properties, the Kuratowski closure axioms:

(KC1) $c(\emptyset) = \emptyset$.
(KC2) For $A \in 2^X$, $A \subseteq c(A)$.
(KC3) For $A \in 2^X$, $c(c(A)) = c(A)$.
For \( A, B \in 2^X \), \( c(A \cup B) = c(A) \cup c(B) \).

These axioms imply:

(KC5) If \( B \subset A \), \( c(B) \subset c(A) \).

Indeed, \( c(A) = c((A \setminus B) \cup B) = c(A \setminus B) \cup c(B) \).

A function \( c : 2^X \to 2^X \) satisfying (KC1)-(KC4) may be called an “abstract closure operator.” Kuratowski noted that any such operator is indeed the closure operator for a topology on \( X \):

**Theorem 1.** (Kuratowski) Let \( c : 2^X \to 2^X \) be an operator satisfying the Kuratowski closure axioms. Then the subsets \( A \in 2^X \) satisfying \( A = c(A) \) obey they axioms (CTS1)-(CTS3) and hence are the closed subsets for a unique topology on \( X \). Moreover \( c \) coincides with closure with respect to this topology.

Proof: Call a set closed if \( A = c(A) \). By (KC1) the empty set is closed; by (KC2) \( X \) is closed. By (KC2) finite unions of closed sets are closed. Now let \( \{A_\alpha\}_{\alpha \in I} \) be a family of closed sets, and put \( A = \cap A_\alpha \). Then for all \( \alpha \), \( A \subset A_\alpha \), so by (KC5), \( c(A) \subset c(A_\alpha) \) for all \( \alpha \), so

\[
c(A) \subset \cap c(A_\alpha) = \cap A_\alpha = A.
\]

Thus the closed sets satisfy (CTS1)-(CTS3) and a topology on \( X \) may be defined by taking the complements of them. We wish to show that \( c(A) = \overline{A} \). We have \( \overline{A} = \cap_{C \supseteq c(A)} A \), the intersection extending over all closed subsets containing \( A \). By (KC3), \( c(A) = c(c(A)) \) is a closed subset containing \( A \) we have \( \overline{A} \subset c(A) \). Conversely, since \( A \subset \cap cC \), \( c(A) \subset \cap cC = \cap cC = \overline{A} \). So \( c(A) = \overline{A} \).

The following result characterizes continuous functions in terms of closure.

**Proposition 2.** (Hausdorff) Let \( f : X \to Y \) be a map of topological spaces. TFAE:

(a) \( f \) is continuous.
(b) For every subset \( S \) of \( X \), \( f(S) \subset \overline{f(S)} \).

2.3. **Interior operator.** The dual notion to closure is the interior of a subset \( A \) in a topological space: \( A^\circ \) is equal to the union of all open subsets of \( A \). In particular a subset is open iff it is equal to its interior. We have

\[
A^\circ = X \setminus \overline{A},
\]

and applying this formula we can mimic the discussion of the previous subsection in terms of axioms for an “abstract interior operator” \( A \mapsto i(A) \), which one could take to be the basic notion for a topological space.

2.4. **Boundary operator.** For a subset \( A \) of a topological space, one defines the boundary

\[
\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap \overline{X \setminus A}.
\]

Alternate terminology: frontier.
Evidently \( \partial A \) is a closed subset of \( A \), and, since \( A = A \cup \partial A \), \( A \) is closed iff \( A \supset \partial A \). A set has empty boundary iff it is both open and closed, a notion which is important in connectedness and in dimension theory.

Example 2.1: Let \( X \) be the real line, \( A = (\infty, 0) \) and \( B = [0, \infty) \). Then 
\[
\partial A = \partial B = \{0\},
\]
and
\[
\partial (A \cup B) = \partial \mathbb{R} = \emptyset 
eq \{0\} = (\partial A) \cup (\partial B);
\]
\[
\partial (A \cap B) = \partial \emptyset = \emptyset 
eq \{0\} = (\partial A) \cap (\partial B).
\]
Thus the boundary operator is not as well-behaved as either the closure or interior operators. We quote from Willard [2, p. 28]: “It is possible, but unrewarding, to characterize a topology completely by its frontier [i.e., boundary] operation.” (I confess that how to do so is not entirely clear to me.)

2.5. Neighborhoods. Let \( x \) be a point of a topological space, and let \( N \) be a subset of \( X \). We say that \( N \) is a neighborhood of \( x \) if \( x \in N \). Open sets are characterized as being neighborhoods of each point they contain.

Let \( N_x \) be the family of all neighborhoods of \( x \). It satisfies the following nice properties:

- (NS1) \( N \in N_x \implies x \in N \).
- (NS2) \( N, N' \in N_x \implies N \cap N' \in N_x \).
- (NS3) \( N \in N_x, N' \supset N \implies N' \in N_x \).
- (NS4) For \( N \in N_x \), there exists \( U \in N_x, U \subset N \), such that \( y \in V \implies V \in N_y \).

Suppose we are given a set \( X \) and a function which assigns to each \( x \in X \) a family \( \mathcal{N}(x) \) of subsets of \( X \) satisfying (NS1)-(NS3). Then the collection of subsets \( U \) such that \( x \in U \implies U \in \mathcal{N}(x) \) form a topology on \( X \). If we moreover impose (NS4), then \( \mathcal{N}(x) = N_x \) for all \( x \).

3. Constructing topological spaces I: topologies on a given set

3.1. Generating sets for topologies; coarser and finer topologies. We begin with the following useful triviality:

Proposition 3. Let \( X \) be a set and \( \{\tau_{\alpha}\}_{\alpha \in A} \) be any collection of topologies on \( X \). Then \( \tau = \cap_{\alpha \in A} \tau_{\alpha} \) is itself a topology on \( X \).

Exercise 3.1: Prove Proposition 3.

Let \( \mathcal{F} \subset 2^X \) be any family of subsets of \( X \), and consider the collection \( \text{Top}(\mathcal{F}) \) of all topologies \( \tau \) on \( X \) such that \( \mathcal{F} \subset \tau \). This collection is nonempty since \( 2^X \) is itself a topology (the discrete topology). Proposition 3 now ensures that \( \text{Top}(\mathcal{F}) \) has a unique minimal element, namely the common intersection of all of its members. We denote this topology by \( \tau(\mathcal{F}) \) and call it the topology generated by \( \mathcal{F} \).

The above argument – namely, that stability under arbitrary intersections leads to a notion of “generation” – is an extremely common one in mathematics. For instance, we have the notion of the subgroup generated by a subset \( S \) of a group.
G, and the notion of the $\sigma$-algebra of subsets generated by an arbitrary collection of subsets $F$ of a set $X$. This notion becomes much more useful if one can describe the “object generated” in some more explicit way: e.g., the subgroup generated by a subset $S$ is just the collection of all group elements of the form $x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$, where $x_i \in S$ and $\epsilon_i \in \{\pm 1\}$; in some informal sense, this is a “one-step process.” In the case of $\sigma$-algebras, it is (unfortunately) the case that the best one can say is that one needs to take countable unions, and then complements, and then countable unions again, and then complements again, and so on, ad (countable) infinitum.

Fortunately, the situation for topologies is much closer to the former example than the latter. Namely, we first form $F_1$ which consists of all finite intersections of elements of elements of $F$; we use here the convention that the empty intersection is all of $X$. We then form $F_2$, which consists of all arbitrary unions of elements of $F_1$; we use here the convention that the empty union is $\emptyset$. Clearly $F_2$ contains $\emptyset$ and $X$ and is stable under arbitrary unions. In fact it is also stable under finite intersections, since for any two families $\{Y_i\}_{i \in I}$, $\{Z_j\}_{j \in J}$ of elements of $F_1$, 

$$\bigcup_i Y_i \cap \bigcup_j Z_j = \bigcup_{i,j} Y_i \cap Z_j,$$

and for all $i$ and $j$, $Y_i \cap Z_j \in F_1$ since $F_\infty$ was itself constructed to be closed under finite intersections. So we are done in two steps: $F_2 = \tau(F)$ is the topology generated by $F$.

Example 3.2: Let $X$ be any nonempty set. If $F = \emptyset$, then $\tau(F)$ is the trivial topology. If $F = \{\{x\} \mid x \in X\}$, $\tau(F)$ is the discrete topology. More generally, let $S$ be any subset of $X$ and $F(S) = \{\{x\} \mid x \in S\}$, then $\tau(S) := \tau(F(S))$ is a topology whose open points are precisely the elements of $S$, so this is a different topology for each $S \subseteq 2^X$.

If $\tau = \tau(F)$, one says that $F$ is a subbasis for $\tau$.

Example 3.3: Let $X$ be a set of cardinality at least 2.

(i) If we take $F$ to be the empty family, then $\tau(F)$ is the indiscrete topology.

(ii) If $Y$ is a subset of $X$ and we take $F = \{Y\}$, then the open sets in the induced topology $\tau_Y$ are precisely those which contain $Y$. Note that these 2$^X$ topologies are all distinct. If $Y = X$ this again gives the indiscrete topology, whereas if $Y = \emptyset$ we get the discrete topology. Otherwise we get a non-Hausdorff topology: indeed for $x \in X$, $\{x\}$ is closed iff $x \in X \setminus Y$. (If $x \in Y$, $\pi = X$.) It is a (rather elementary) exercise in set theory to show that $\tau_Y$ and $\tau_Y'$ are homeomorphic spaces iff $Y$ and $Y'$ have the same cardinality.

The nomenclature “subbasis” suggests the existence of a cognate concept, that of a “basis”. If $F$ is a subbasis, it would be natural to guess that $F_1$ is a basis, or more precisely that a basis for a topology should be a collection of open sets, closed under finite intersection, whose unions recover all the open sets. But it turns out that a weaker concept is much more useful.

Definition: A family of subsets $B$ of a set $X$ is called a basis if

(B1) $\forall U_1, U_2 \in B$ and $x \in U_1 \cap U_2$, $\exists U_3 \in B$ such that $x \in U_3 \subset U_1 \cap U_2$; and
(B2) For all \( x \in X \), there exists \( U \in \mathcal{B} \) such that \( x \in U \).

The point is that, for the purposes of defining a topology, this is just as good as being closed under finite intersections:

**Proposition 4.** The topology generated by a basis \( \mathcal{B} \) is precisely the set of arbitrary unions of elements of \( \mathcal{B} \).

**Exercise 3.4:** Prove Proposition 4.

**Example 3.5:** In a metric space \((X, d)\), then open balls form a basis for the topology: especially, the intersection of two open balls need not be an open ball but contains an open ball about each of its points. Indeed, the open balls with radii \( \frac{1}{n} \), for \( n \in \mathbb{Z}^+ \), form a basis.

**Example 3.6:** In \( \mathbb{R}^d \), the \( d \)-fold products \( \prod_{i=1}^d (a_i, b_i) \) of open intervals with rational endpoints is a basis. In particular this shows that \( \mathbb{R}^d \) has a *countable basis*, which will turn out to be an extremely nice property for a topological space to have.

The set of all topologies on a set \( X \) is partially ordered by inclusion.\(^5\) If there are two topologies \( \tau_1 \) and \( \tau_2 \) on a set \( X \), then all of the following are used to mean that \( \tau_1 \subset \tau_2 \):

- \( \tau_2 \) is finer than \( \tau_1 \).
- \( \tau_1 \) is coarser than \( \tau_2 \).
- \( \tau_2 \) is stronger than \( \tau_1 \).
- \( \tau_1 \) is weaker than \( \tau_2 \).\(^6\)

**Examples 3.7:** The trivial topology is the coarsest topology on \( X \) and the discrete topology is the finest topology on \( X \). The cofinite topology on the complex plane is strictly weaker than the Euclidean topology, a fact which is well appreciated by algebraic geometers.

**Exercise 3.8 (Sorgenfrey line):** On \( \mathbb{R} \), show that intervals of the form \([a, b)\) form a basis for a topology which is strictly finer than the Euclidean topology on \( \mathbb{R} \).

A subset \( S \) of a topological space \( X \) is said to be *dense* if \( \overline{S} = X \); equivalently, every nonempty open subset meets \( S \). A space is said to be *separable* (a terrible piece of terminology hallowed by time!) if it has a countable dense subset.

**Exercise 3.9:**

a) Show that a space with a countable basis has a countable dense subset.

b) Show that the converse holds if \( X \) is metrizable.

\(^5\)Indeed, it is a complete lattice.

\(^6\)If we think of the case of metric spaces where open sets are unions of “pebbles” – i.e., open balls – then the terms coarser and finer are quite natural. The terms stronger and weaker take some getting used to. Once we introduce convergence, we will immediately see that to say that if \( \tau_1 \subset \tau_2 \), a sequence (or a net!) which converges in \( \tau_2 \) necessarily also converges in \( \tau_1 \), so an assertion of convergence in a stronger topology is a stronger assertion. As a student I was warned that some people use stronger and weaker with exactly the opposite meaning, but I must say that I have never encountered such a person.
Example 3.10 (Moore plane continued): The Moore plane is a Hausdorff space. It is also separable: indeed, take the set of pairs \((x, y)\) with \(x \in \mathbb{Q}, y \in \mathbb{Q}^+\). However it is not second-countable. Indeed, for each \(x \in \mathbb{R}\), consider the basic open set \(U_x := B((x, 0), 1) \cup (x, 0)\). If there were a countable basis, there would exist a countable collection of open sets \(V_n\) such that every \(U_x\) contains some \(V_n\), but if \((x, 0) \in V_n \subset U_x\), then \(V_n\) does not contain \((x', 0)\) for any \(x' \neq x\). Since the real line is uncountable, this is a contradiction.

3.2. Neighborhood bases. Let \(x\) be a point of a topological space \(X\). A family \(\{N_\alpha\}\) of neighborhoods of \(x\) is said to be a neighborhood base at \(x\) (or a fundamental system of neighborhoods of \(x\)) if every neighborhood \(N\) of \(x\) contains some \(N_\alpha\).

Suppose given a neighborhood basis \(N_x\) at \(x\) for all \(x \in X\). The following axioms hold:

\begin{align*}
(NB1) & \quad N \in B_x \implies x \in N. \\
(NB2) & \quad N, N' \in B_x \implies \text{there exists } N'' \in B_x \text{ such that } N'' \subset N \cap N'. \\
(NB3) & \quad N \in B_x \implies \text{there exists } V \in B_x, V \subset N, \text{ such that } y \in V \implies V \in B_y.
\end{align*}

Conversely:

**Proposition 5.** Suppose given a set \(X\) and, for each \(x \in X\), a collection \(B_x\) of subsets satisfying (NB1)-(NB3). Then the collections \(N_x = \{Y \mid \exists N \in B_x | Y \supset N\}\) are the neighborhood systems for a unique topology on \(X\), in which a subset \(U\) is open iff \(x \in U \implies U \in N_x\). Each \(N_x\) is a neighborhood basis at \(x\).

Exercise 3.11: Prove Proposition 5.

Remark: Consider the condition

\(\text{(NB3')} \quad N \in B_x, y \in N \implies y \in N.\)

Replacing (NB3) with (NB3’) amounts to restricting attention to open neighborhoods. Since (NB3’) \(\implies\) (NB3), we may specify a topology on \(X\) by giving, for each \(x\), a family \(N_x\) of sets satisfying (NB1), (NB2), (NB3’). This is a very convenient way to define a topology: e.g. the metric topology is thus defined just by taking \(N_x\) to be the family \(\{B(x, \epsilon)\}\) of \(\epsilon\) balls about \(x\).

**Proposition 6.** Suppose that \(\varphi : X \to X\) is a self-homeomorphism of the topological space \(x\). Let \(x \in X\) and \(N_x\) be a neighborhood basis at \(x\). Then \(\varphi(N_x)\) is a neighborhood basis at \(y = \varphi(x)\).

Proof: It suffices to work throughout with open neighborhoods. Let \(V\) be an open neighborhood of \(y\). By continuity, there exists an open neighborhood \(U\) of \(x\) such that \(\varphi(U) \subset V\). Since \(\varphi^{-1}\) is continuous, \(\varphi(U)\) is open.

Observe that (as in any category) the automorphisms of a topological space \(X\) form a group, \(\text{Aut}(X)\). We say \(X\) is homogeneous if \(\text{Aut}(X)\) acts transitively on \(X\), i.e., for any \(x, y \in X\) there exists a self-homeomorphism \(\varphi\) such that \(\varphi(x) = y\). By the previous proposition, if a space is homogeneous we can recover the entire topology from the neighborhood basis of a single point. In particular this will apply to topological groups.