LECTURES ON PERIOD-INDEX PROBLEMS (SHORT VERSION)

PETE L. CLARK

These are my notes for an April 24, 2006 lecture given at the Mathematical Sciences Research Institute on the subject of period-index problems. This is true in a literal sense: there is another, significantly more detailed, writeup available which was pared down to yield the current version. (So if you are seriously interested in these matters, you will want to be reading the other version instead.)

Let K be a field of characteristic $\mathrm{char}(K)$, with *separable* algebraic closure $\overline{K}$ and absolute Galois group $\mathfrak{g}_K$. We assume that all varieties over K are nonsingular, projective and geometrically integral.

To simplify matters, let us only work with positive integers P which are prime to the characteristic of K.

1. The two kinds of period-index problems

1.1. **Cohomological period-index problems.** Let G be a $\mathfrak{g}_K$-module (i.e., a *commutative* group endowed with an action of $\mathfrak{g}_K$ continuous for the discrete topology on G and the profinite topology on $\mathfrak{g}_K$), i a positive integer, and $\eta \in H^i(K,G)$ a Galois cohomology class. Define respectively the **period** and **index**

$$P(\eta) = \#\langle \eta \rangle.$$
$$I(\eta) = \gcd\{[L:K] \mid \eta|_L = 0\}.$$

Fact 1.
a) $P \mid I$.
b) $I \mid P^\infty$ (i.e., $\exists\ \alpha$ such that $I \mid P^\alpha$.)

Proof: This is classical; see e.g. [3, Prop. 11].

The period-index problem is then: what can be said about the α (for fixed η, as η varies in a fixed group $H^i(K,G)$, or as G varies in some family)? Especially, when can we take $\alpha = 1$?

$$I = 1 \iff P = 1 \iff \eta = 0. \quad (1)$$

Proposition 1. *(Lenstra, [3, Prop. 12]) Suppose G is finite, and $\eta \in H^1(K,G)$. Then $I(\eta) \mid \#G$.*

Example 1.1.1: The group $H^1(K,G)$ parameterizes torsors X under G in the category of varieties. We will shortly be looking at the case of G an abelian variety.

Example 1.1.2: $H^2(K,\mathbb{G}_m) = \mathrm{Br}(K)$, the Brauer group of K. Here $I(\eta)$ is interpreted as $\sqrt{[D:K]}$, where D is the unique division algebra representative of η, and P is the least n such that $D^{\otimes n}$ is a matrix algebra.

Definition: For a non-negative integer α, we say that a field K has property $\mathrm{Br}(\alpha)$ if for any finite extensions L/K and any $\eta \in \mathrm{Br}(L)[P]$, $I(\eta) \mid P^\alpha$. (We will say that a field has property $\mathrm{Br}(-1)$ if it is separably closed.)

Fact 2. *The following fields have the property* $\mathrm{Br}(0)$*:*
a) $\mathbb{F}_p$.
b) $\overline{k}(t)$.
c) A complete discretely valued field (CDVF) with algebraically closed residue field.
d) A pseudoalgebraically closed (PAC) field.

Fact 3. *The following fields have the property* $\mathrm{Br}(1)$*:*
a) $\mathbb{R}$.
b) A CDVF whose residue field is $\mathrm{Br}(0)$ *(e.g.* $\mathbb{Q}_p$, $\mathbb{F}_p((t))$*).*
c) $\mathbb{Q}$, $\mathbb{F}_p(t)$.
d) $\overline{k}(t_1, t_2)$.

Summary: For any Galois cohomology class, we have a well-defined period and index. The former quantity seems more natural, and the latter seems more mysterious.

1.2. **Geometric period-index problems.**
Let $V_{/K}$ be a variety. We define its **index** $I(V)$ to be the least positive degree of a K-rational 0-cycle. Equivalently, it is the cardinality of the cokernel of the map $\deg : CH_0(X) \to \mathbb{Z}$, and also the gcd of $[L : K]$ such that $V(L) \neq \emptyset$.

To define the period, we need to recall that there exists a variety $\mathbf{Alb}^1(V)$ which is a torsor under a abelian variety $\mathbf{Alb}^0(V)$, and a morphism $V \to \mathbf{Alb}^1(V)$ which is universal for morphisms into abelian torsors. In particular, $\mathrm{Alb}^1(V)$ corresponds to a class $\eta_V \in H^1(K, \mathbf{Alb}^0(V))$, and we define the **period** of $P(V)$ to be the period of $P(\eta_V)$.

Example 1.2.1: Let V be a Severi-Brauer variety, so $[V] \in \mathrm{Br}(K)$. Then $I(V) = I([V])$, but since V is simply connected, $\mathbf{Alb}^0(V) = 0$ and $P(V) = 1$ (so is usually not equal to $P([V])$).

Example 1.2.2: Let V be a curve, so Alb = Pic. Then the index of V is the least positive degree of a K-rational divisor and the period is the least positive degree of a K-rational divisor class (a significantly more enlightening definition than in the general case). We have that the period of V is the period of $\mathbf{Pic}^1(V)$. Since we have a morphism $V \to \mathbf{Pic}^1(V)$, we clearly have

$$I(\mathbf{Pic}^1(V)) \mid I(V).$$

A very important open problem is to understand the discrepancy $I' = I(V)/I(\mathbf{Pic}^1(V))$ between these two indices.

Example 1.2.3: Consider a genus zero curve $V_{/K}$. If $V(K) = \emptyset$, $I(V) = 2$, whereas $P(V) = 1$ implies $I(\mathbf{Pic}^1(V)) = 1$, i.e., the two indices differ by a factor of 2.

Open Problem 1. *What are the possible values of $I'(C)$ for a curve of genus g? Can we have $I'(C) > 2$?*

Moral: the geometric and cohomological period-index problems are distinct but closely related, and it would be of interest to understand the relationships between them more clearly.

In what remains, we will consider the case in which the two problems are the same, namely:

Example 1.2.4: Suppose that $X \mapsto \mathrm{Alb}^1(X)$ is an isomorphism, i.e., X is a torsor under the abelian variety $\mathbf{Alb}^0(X)$.

In light of all of this, the best case scenario is when X simultaneously a torsor under an abelian variety and a curve, i.e., is a curve of genus one.

2. Curves of genus one

2.1. Generalities.

Let $C_{/K}$ be a genus one curve, with Jacobian elliptic curve E.

Proposition 2. *For all curves of genus one, $I \mid P^2$.*

This follows easily (under our simplifying assumptions on P and $\mathrm{char}(K)$) from the Kummer sequence

$$0 \to E(K)/PE(K) \to H^1(K, E[P]) \to H^1(K, E)[P] \to 0, \tag{2}$$

a point which we will revisit later in more detail.

Theorem 3. *(Lang-Tate)*
Let E be an elliptic curve over $\mathbb{C}((t_1))((t_2))$ obtained from an elliptic curve over $\mathbb{C}$. Then for $P \mid I \mid P^2$, there exists $\eta \in H^1(K, E)$ with period P and index I.

In some sense, this gives an answer to the period-index problem for genus one curves. To say more, we need to make some assumptions on the field.

2.2. Some fields with $P = I$.

Example 2.1: K algebraically closed, K PAC, K finite. Indeed then all WC-groups are trivial (which was shown in a previous lecture). Note that it follows from the Weil bounds that $I(V) = 1$ for all $V_{/\mathbb{F}_q}$, and this gives another proof.

Theorem 4. *([5], [7]) Suppose* $\mathrm{Br}(K) = 0$*. Then $I = P$ for* all *curves $C_{/K}$.*

Proof: We must show that if D is a rational divisor class on C, there is some other rational divisor class of the same degree that is represented by a rational divisor. In fact, if $\mathrm{Br}(K) = 0$ we have that every rational divisor class is represented by a rational divisor. For this, we use the following fundamental sequence

$$0 \to \mathrm{Pic}(V) \to \mathbf{Pic}(V)(K) \xrightarrow{\delta} \mathrm{Br}(K) \xrightarrow{\gamma} \mathrm{Br}(V). \tag{3}$$

Thus given a rational divisor class D, there is a well-defined element $\delta(D) \in Br(K)$ whose nontriviality is precisely the obstruction to D being represented by a divisor. The result is now clear.

The following result exploits a similar idea.

Theorem 5. *(Olson, C—) Let K be a global field, and $C_{/K}$ a curve which has points everywhere locally except possibly at one place of K. Then $P(C) = I(C)$.*

The proof is an exercise.

Theorem 6. *(Lichtenbaum) Let K be a locally compact field. Then $I = P$ for genus one curves $C_{/K}$.*

It would hardly be overstating things to say that all subsequent work on geometric period-index problems builds on the ideas behind this theorem in some way.

Theorem 7. *(C—, '04)*
a) Let K be a number field, and $E_{/K}$ an elliptic curve with $E(K) = 0$. Then for all $n \in \mathbb{Z}^+$, there exists $\eta \in H^1(K, E)$ with $P(\eta) = I(\eta) = n$.
b) If moreover $Ш(K, E) = 0$, then for any finite L/K and any $n \in \mathbb{Z}^+$, there exists $\eta \in H^1(K, E)$ such that $\eta|_L$ has period equals index equals n.

Remark X.X.X: There *do* exist elliptic curves $E_{/\mathbb{Q}}$ with $E(\mathbb{Q}) = Ш(\mathbb{Q}, E) = 0$, as was shown by Kolyvagin.

Remark X.X.X: Earlier, W. Stein had shown that over any number field there exists a curve of any given index which is not divisible by 8.

Let us sketch the proof of part a). If $E(K) = 0$, one can produce for all n a class $\eta \in H^1(K, E)$ which is locally trivial except at a single place of K and has local period n at that place. One reduces to the case $n = p^a$, and the key observation is that *either* $Ш(K, E)$ is p-divisible – in which case Theorem 5 implies there exist elements of $P = I = p^a$ for all a – or $Ш(K, E)[p^\infty]$ is finite, in which case one can apply the duality theory of Poitou-Tate to construct such a class. By Theorem 5, every multiple of η has period equals index, and clearly some multiple of η has period n.

2.3. $P < I$ over Global Fields.

Cassels produced an example of a genus one curve $C_{/Q}$ with $P = 2,\ I = 4$. In his example, $E(\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. The following is therefore a generalization:

Theorem 8. *(C— '03, Sharif '06) Let K be a global field, and p a prime such that $E(K)[P] \cong (\mathbb{Z}/P\mathbb{Z})^2$. For any $P \mid I \mid P^2$, there exist infinitely many classes of period P and index I.*

Remarks: The case of prime P was established in [**?**], whereas the general case appears in Sharif's 2006 Berkeley thesis. It seems to me that the methods of [**?**] could also be used to establish the general case. This would, however, require replacing a messy Hilbert symbol calculation by a *messier* Hilbert symbol calculation, whereas Sharif's approach exploits work of Lichtenbaum to proceed in a much more elegant way. For instance, his construction produces classes which are nontrivial at exactly two places of K, which serves as a sort of converse to Theorem 5.

Theorem 9. *(Sharif '06) For any odd P, there exist genus one curves $C_{/\mathbb{Q}}$ with period P and index P^2.*

It seems to me that, building on the ideas of this theorem, one should be able to prove the following stronger result:

Theorem? 10. *Let K be a global field, P a positive integer not divisible by* $\mathrm{char}(K)$ *and $E_{/K}$ an elliptic curve. Then there are infinitely many classes in $H^1(K,E)$ of period P and index P^2.*

Ideally, one would like to establish the following result:

Conjecture 11. *Let K be an infinite, finitely generated field and $E_{/K}$ an elliptic curve. Then for all $P \mid I \mid P^2$, there exist infinitely many classes $\eta \in H^1(K,E)$ of period P and index I.*

2.4. **Some remarks in the way of proof.** I. $H^1(K,E[P])$ classifies pairs (C,D), where $C \in H^1(K,E)$ and D is a degree P rational divisor class on C modulo equivalence: $(C,D) \sim (C',D')$ if there exists an isomorphism of torsors $f : C \to C'$ such that $f^*(D') = D$.

We can thus reinterpret the Kummer sequence

$$0 \to E(K)/PE(K) \to H^1(K,E[P]) \to H^1(K,E)[P]ra0$$

as $(C,D) \mapsto C$. Accordingly we may define

$$\Delta : H^1(K,E[P]) \to \mathrm{Br}(K)$$

by $(C,D) \mapsto \delta(D)$.

II. $\mathrm{Im}(\Delta) \subset \mathrm{Br}(K)[P]$.
This is a key point that we will return to in the higher-dimensional case. For $g = 1$ it was first proved by Lichtenbaum, with a different proof by O'Neil. Note that we are claiming more than that the image is contained in $\mathrm{Br}(K)[I]$, which is easily seen to be true for all varieties.

III. Put $G = \iota(E(K)/PE(K)) \subset H^1(K,E[P])$. In all cases of interest, this group if finite. So pick a lift $\xi \in H^1(K,E[P])$ of η. Then $P(\eta) = N(\eta)$ iff: there exists $g \in G$ such that $\Delta(g\xi) = 0$.

Can also show: if for all $g \in G$, $\#\Delta(g\xi) = R$, then $I(\eta) = P \cdot R$.

Special case: $E(K)/PE(K) = 0$. Then the lift ξ is unique, and $I = P \cdot \#\Delta(\xi)$.

IV. If $\#E(K)[P] = P^2$,

$$H^1(K,E[P]) \cong (K^\times/K^{\times P})^2,$$

and (more or less) Δ is given as the norm-residue (or "Hilbert") symbol $\langle a,b\rangle$.

Thus we can prove Theorem 8 "without looking at G" but essentially by solving a Hilbert symbol difference equation.

V. Suppose K is locally compact, so $\mathrm{Br}(K)[P]$ is cyclic of order P. Also $\#G = \#H^1(K,E)[P]$. We can define a map

$$L : H^1(K,E[P]) \times E(K)/PE(K) \to \mathrm{Br}(K),$$

by

$$L(\xi,Q) := \Delta(\xi + Q) - \Delta(\xi) - \Delta(Q)$$

$$= \Delta(\xi + Q) - \Delta(\xi).$$

In fact this clearly descends to a pairing on $H^1(K,E)[P] \times E(K)/PE(K)$.

Lichtenbaum showed: $L = T$, Tate's duality pairing.

In particular, when K is locally compact, L is perfect, and Lichtenbaum's theorem follows easily.

3. Torsors under abelian varieties

Let $A_{/K}$ be a g-dimensional abelian variety.

Proposition 12. *For all classes $\eta \in H^1(K,A)$, $I \mid P^{2g}$.*

Exercise X.X.X: Let $K_g = \mathbb{C}((t_1)) \cdots ((t_{2g}))$. Show that for all $P \mid I \mid P^{2g}$, there exists a g-dimensional abelian variety $A_{/K_g}$ and a class $\eta \in H^1(K_g, A)$ of period P and index I.

Note that K_g is quite a complicated field. In the case of $g = 1$, we saw that such esoterica was unnecessary, in that all possible values of P and I arise over suitable number fields. In higher dimensions, this is not at all the case.

In all the remaining results of the section, we assume that A possesses a **principal bundle**, i.e., an ample line bundle λ such that the induced map

$$\Phi_L : A \to A^\vee,\ x \mapsto \tau_x^* L \otimes L^{-1},$$

is an isomorphism.[1]

Theorem 13. *If K has property $\mathrm{Br}(\alpha)$, then for $X \in H^1(K,A)[P]$,*

$$M(X) \leq 2^\alpha g! P^{g+\alpha}.$$

Theorem 14. *Let K be a locally compact field. Then*

$$M(X) \leq 2g! P^g.$$

Supplement: In the setting of Theorems 13 and 14, assume **any** of the following additional hypotheses:
a) $g = 1$.
b) P is odd.
c) $A[P]$ is isomorphic as a $\mathfrak{g}_K$-module to $H \oplus \mathrm{Hom}(H, \mathbb{G}_m)$.
Then we may replace $2g!$ by $g!$.

Remark: It is known that for K a sufficiently large p-adic field (depending on g), there exist g-dimensional torsors of period p and index p^g.

Theorem 15. *Suppose that K is p-adic, $NS(A)$ is cyclic, $X \in H^1(K,A)$ of period P, and that at least one of the hypotheses a)-c) is satisfied. Then the Brauer kernel*

$$\kappa(X) = \mathrm{Ker}(\mathrm{Br}(K) \to \mathrm{Br}(X))$$

is cyclic of order P.

[1]This is subtly stronger than saying "Let $(A, \lambda)_{/K}$ be a principally polarized abelian variety." We will not pause to discuss the difference, rather referring the reader to [**?**] for the full story. However, when K is locally compact, there is no distinction.

Remark: This is to be contrasted with the case of curves over p-adic fields, where the Brauer kernel has equal order to the index of C. (Recall that the period and index need not coincide in either case.)

Definition: For a variety $V_{/K}$ the **Picard index** $I_{\mathrm{Pic}}(V)$ is the exponent of the cokernel of the natural map $\mathbf{Pic}(V)(K) \to NS(V)(K)$.

Then, under the hypotheses of Theorem 15, we are showing that the period is equal to the Picard index.

Open Problem 2. *Let $V_{/K}$ be a variety over a p-adic field. Is it the case that*

$$I_{\mathrm{Pic}}(V) = \#\kappa(V/K)?$$

What if we assume moreover that $NS(V)(K)$ is cyclic?

Remark: The answer is affirmative for quadric hypersurfaces.

3.1. **Some proofs.** In what follows, we fix $L = P\lambda$, where λ is our principal line bundle on A. Associated to the finite morphism $\Phi_L : A \to A^\vee$, we get a Kummer sequence

$$A^\vee(K)/PA(K) \to H^1(K, A[P]) \to H^1(K, A)[P] \to 0.$$

Let $X \in H^1(K, A)[P]$. It is not hard to show that $NS(X)$ is canonically isomorphic, as a $\mathfrak{g}_K$-module to $NS(A)$. We can thus speak of $\mathbf{Pic}^\lambda(X)$, the set of rational divisor classes on X which are algebraically equivalent to λ.

We now have the following result, which directly generalizes the one-dimensional case:

Proposition 16.
a) The group $H^1(K, E[P])$ parameterizes equivalence classes of pairs (X, D), where $X \in H^1(K, A)$ and $D \in \mathbf{Pic}^\lambda(X)(K)$.
b) The forgetful map $(C, D) \mapsto C$ corresponds to the map $H^(K, A[P]) \to H^1(K, A)[P]$ in Galois cohomology, and its kernel, namely the equivalence classes of such lines bundles on A itself, is identified with $A^\vee(K)/PA^\vee(K)$.

Thus as before we may define the **period-index obstruction map**

$$\Delta = \Delta_L : H^1(K, A[P]) \to \mathrm{Br}(K),$$

by

$$(X, D) \mapsto \delta(D).$$

Notice, however, that in contrast to the one-dimensional case, it is not immediately clear what this divisorial construction has to do with the index of V, a quantity defined in terms of zero-cycles. (Of course it is not so farfetched that on a principlally polarized abelian variety, there should be some connection....) To see the relevance, we need another interpretation of $H^1(K, A[P])$:

Proposition 17. *The group $H^1(K, A[P])$ classifies equivalence classes of "diagrams" $X \to V$ which are twisted forms of $\varphi_L : A \to \mathbb{P}^{P^g-1}$. Two diagrams are regarded as equivalent if they fit into a commutative square*

INSERT.

We can pass directly from the first interpretation to the second by noticing that V is the Severi-Brauer variety associated to the ample, basepointfree divisor class D on X. The key point is thus that, whereas on a variety over an algebraically closed field, an ample basepointfree divisor class corresponds to a (projective equivalence class of) morphisms into projective space, on an arbitrary variety, such a class corresponds to a morphism into a Severi-Brauer variety.

But this is an important distinction: given an embedding into projective space, we can take a hyperplane section to recover the divisor class. On the other hand, we can intersect with a lower-dimensional linear subspace to get a K-rational effective *zero-cycle* on X. The order of this zero-cycle is precisely the **degree** of the morphism. The degree of a morphism is of course a geometric property ("numerical properties are geometric"), so the degree is equal to the degree of the morphism $\varphi_L : A \to \mathbb{P}^{P^g-1}$, which is well-known (Riemann-Roch) to be $g!P^g$. Thus we get the following:

Theorem 18. *Let* $\eta_X \in H^1(K,E)[P]$. *Suppose there exists a Kummer lift* ξ *of* η *with* $\Delta(\xi) = 0$. *Then* $M(\eta) \leq g!P^g$.

The next step is an analogue of Proposition **??**. We begin with the following:

Proposition 19. *Assume that* P *is odd. Then* $\mathrm{Im}(\Delta) \subset \mathrm{Br}(K)[P]$.

Notice that the proof given in the one-dimensional case does not work: the Severi-Brauer varieties which intervene are now of dimension $P^g - 1$, so that the bound on the *index* of the Brauer classes that we get is P^g: no good. (In fact, in [3] I showed that this bound is sharp for certain fields, namely those for which the period-index discrepancy in $\mathrm{Br}(K)$ is sufficiently large.)

On the other hand, O'Neil's interpretation of Δ in terms of nonabelian Galois cohomology of theta groups immediately gives, using a general theorem of Zarhin, the statement that Δ is a **quadratic** map, i.e.,

$$L(\xi,\psi) := \Delta(\xi+\psi) - \Delta(\xi) - \Delta(\psi).$$

Unfortunately quadratic maps between abelian groups need not preserve orders of elements: you can check that the map from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/4\mathbb{Z}$ which sends 0 to 0 and 1 (mod 2) to 2 (mod 4) is quadratic! But in fact this is essentially "as bad as it gets":

Proposition 20. *Let* $f : X \to Y$ *be a quadratic map between abelian groups, with* $f(0) = 0$. *Then:*
a) $f(X[P]) \subset Y[2P]$.
b) For odd P, $f(X[P]) \subset Y[P]$.

In fact there is another way of seeing the quadraticity of Δ which avoids Galois cohomology of theta groups (which we do not consider in these notes). Namely, as before, $L(\xi,\psi)$ descends to a map

$$L : A^\vee(K)/PA^\vee(K) \times H^1(K,A)[P] \to \mathrm{Br}(K)$$

and now a miracle recurs:

Theorem 21. *The map* L *is nothing else than Tate's duality map applied to the dual variety* $A^\vee$. *(In particular it is bilinear!)*

As yet the proof of this theorem has not been written down. Rather, as a consequence of his more elaborate *pseudo-motivic homology*, he derives:

Corollary 22. *For any locally compact field K, and $X \in H^1(K, A)$ of period P,*

$$\kappa^0(V/K) = \delta(\mathbf{Pic}^0(V)(K)) = \mathrm{Br}(K)[P].$$

Quite recently, van Hamel was kind enough to sketch a proof of the theorem itself. His proof is too "high tech" (in particular, it takes place in the derived category) to be included here.[2]

Anyway, as in the one-dimensional case, the corollary is exactly what we need, together with the inclusion $\kappa(V/K) \subset \mathrm{Br}(K)[P]$, to conclude that we can modify any given Kummer lift of V by a divisor class algebraically equivalent to zero to get one with vanishing obstruction.

What about when P is even (and $g > 1$)? In fact, we are able to prove that the image of the obstruction is still contained in $\mathrm{Br}(K)[P]$ provided we have the existence of a **Lagrangian decomposition** $A[P] \cong H \oplus H^*$, so in particular when we have full level structure. The proof of this requires more work – namely, an investigation into the Galois cohomology of Heisenberg-type group schemes – so we shall say nothing about it here.

Finally, if we assume that the Néron-Severi group is generated by λ, then $\kappa(V/K)$ is generated by $\kappa^0(V/K)$ together with $\Delta(\xi)$ for any Kummer lift ξ of η, so combining with Tate-Lichtenbaum-van Hamel duality, we get that

$$\kappa(V/K) = \mathrm{Br}(K)[P].$$

References

[1] P.L. Clark. *Period-index problems in WC-groups I: elliptic curves*, J. Number Theory, 2005.

[2] P.L. Clark, *There are genus one curves of every index over every number field*, to appear in J. Reine Angew. Math.

[3] P.L. Clark, *Period-index problems in WC-groups II: abelian varieties*, submitted.

[4] S. Lang and J. Tate, *Principal homogeneous spaces over abelian varieties*, Amer. J. Math. 80 (1958), 659-684.

[5] S. Lichtenbaum, *The period-index problem for elliptic curves*, Amer. J. Math. 90 (1968), 1209-1223.

[6] S. Lichtenbaum, *Duality theorems for curves over p-adic fields*, Invent. Math. 7 (1969), 120-136.

[7] L. Olson, Amer. J. (1970).

The Mathematical Sciences Research Institute, 17 Gauss Way, Berkeley, CA 94709

E-mail address: plclark@msri.org

[2] I would be very happy to receive a writeup for a later edition of these notes. . .