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# A Proof of the Existence of Infinite Product Probability Measures

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Sadahiro Saeki

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*In memory of my dear friend Karl Stromberg*

Let  $\{(\Omega_i, \mathcal{F}_i, P_i): i \in I\}$  be a nonempty collection of probability spaces, and let  $\Omega := \prod_i \Omega_i$  be the product space. A measurable cylinder in  $\Omega$  is a subset  $A$  of  $\Omega$  of the form  $A = \prod_i A_i$ , where  $A_i \in \mathcal{F}_i$  for each  $i$  and  $A_i = \Omega_i$  for all but finitely many  $i$ 's. For such a set  $A$ , define  $P(A) := \prod_i P_i(A_i)$ . By definition, the product probability measure of the  $P_i$ 's is the (necessarily unique) extension of  $P$  to a probability measure on  $\mathcal{F}(\mathcal{M}c)$ , where  $\mathcal{M}c$  is the collection of all measurable cylinders in  $\Omega$  and  $\mathcal{F}(\mathcal{M}c)$  is the  $\sigma$ -field generated by  $\mathcal{M}c$ . The standard proof of the existence of the product probability measure is based upon Fubini's Theorem for finitely many factors; see [HS: pp. 429–435]. We give a simple proof that does not require Fubini's Theorem.

**Lemma.** *Let  $\mu: \mathcal{M}c \rightarrow [0, 1]$  be a function such that  $\sum_1^\infty \mu(A_n) = 1$  whenever  $(A_n)$  is a disjoint sequence in  $\mathcal{M}c$  with union  $\Omega$ . Then  $\mu$  extends uniquely to a probability measure on  $\mathcal{F}(\mathcal{M}c)$ .*

*Proof:* Let  $\mathcal{D}$  be the collection of all finite unions of measurable cylinders. It is easy to check that  $\mathcal{D}$  is a field and each  $A \in \mathcal{D}$  can be written as a finite disjoint union of members of  $\mathcal{M}c$ . In particular,  $A$  can be written as a countable disjoint union of members of  $\mathcal{M}c$ , say  $A = \bigcup_1^\infty A_n$ . Let  $\mu'(A) := \sum_1^\infty \mu(A_n)$ . To see that  $\mu'$  is well-defined, write  $\Omega \setminus A = \bigcup_1^m B_k$  with pairwise disjoint  $B_k \in \mathcal{M}c$ . Then

$$\sum_1^\infty \mu(A_n) = 1 - \sum_1^m \mu(B_k) \tag{1}$$

by our assumption on  $\mu$ . Since the right-hand of (1) has nothing to do with the decomposition  $\bigcup_1^\infty A_n$  of  $A$ , it follows that  $\mu'$  is well-defined and therefore countably additive of  $\mathcal{D}$ . Hence the desired result is an immediate consequence of E. Hopf's extension theorem [HS: p. 142]. ■

**Theorem.**  *$P$  extends uniquely to a probability measure on  $\mathcal{F}(\mathcal{M}c)$ .*

*Proof:* It suffices to prove that  $P$  satisfies the hypothesis of the lemma. Without loss of generality, assume that  $I$  is an infinite set. Let  $(A_n)$  be a disjoint sequence in  $\mathcal{M}c$  with union  $\Omega$ .

*Case 1:*  $I$  is countable. Then we may assume  $I = \mathbb{N}$ . Write  $A_n = \prod_{i=1}^\infty A_{n,i}$ , where  $A_{n,i} \in \mathcal{F}_i$  for each  $i$  and  $A_{n,i} = \Omega_i$  for all  $i > i_n \in \mathbb{N}$ . We claim that if  $m \in \mathbb{N}$  and  $x = (x_i)$  is an element of  $A_m$  and if  $n \in \mathbb{N}$ , then

$$\left\{ \prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i) \right\} \prod_{i>i_m} P_i(A_{n,i}) = \delta_{m,n} \quad (\text{Kronecker's delta}). \tag{2}$$

For  $n = m$ , this is trivial, so assume  $n \neq m$ . Then, since  $\sum_1^\infty \chi_{A_k} = 1$  identically and  $\chi_{A_m}(x_1, \dots, x_{i_m}, y_{i_m+1}, \dots) = 1$  for all  $y_i \in \Omega_i$  with  $i > i_m$ , we have

$$\left\{ \prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i) \right\} \prod_{i>i_m} \chi_{A_{n,i}}(y_i) = 0 \quad (3)$$

for all such  $y_i$ 's. Integrating each side of (3) finitely many times, we obtain (2) for  $n \neq m$ .

To get a contradiction, suppose  $\sum_{n=1}^\infty P(A_n) \neq 1$ . Then there must exist an  $x_1 \in \Omega_1$  such that

$$\sum_{n=1}^\infty \chi_{A_{n,1}}(x_1) \prod_{i=2}^\infty P_i(A_{n,i}) \neq 1.$$

Hence an inductive argument yields an element  $x = (x_i)$  of  $\Omega$  such that

$$\sum_{n=1}^\infty \left\{ \prod_{i=1}^k \chi_{A_{n,i}}(x_i) \right\} \prod_{i=k+1}^\infty P_i(A_{n,i}) \neq 1 \quad (4)$$

for each  $k \geq 1$ . But  $x \in A_m$  for some  $m \in \mathbb{N}$ . Hence (4) with  $k = i_m$  contradicts (2).

*Case 2:  $I$  is uncountable.* Then we can choose a countable subset  $J$  of  $I$  such that  $A_n = A'_n \times \Omega'$  for all  $n \geq 1$ , where each  $A'_n$  is a measurable cylinder in  $\prod_{i \in J} \Omega_i$  and  $\Omega' = \prod_{i \notin J} \Omega_i$ . By Case 1 applied to  $(A'_n)$ , we obtain  $\sum_1^\infty P(A_n) = 1$ . ■

*Dedication. Professor Karl Stromberg, my friend and colleague, died on July 3, 1994. He was an enthusiastic lover of the Monthly. When I presented the above proof in my seminar five to eight years ago, he liked it very much. Karl, I dedicate the present paper to you in the memory of our friendship. Have a peaceful sleep!*

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#### REFERENCE

[HS] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, Berlin, 1965.

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## A Problem

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Leo S. Gurin

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**TRIBUTE.** I learned about this problem and its solution in 1935, when I was in the eighth grade, from my teacher of mathematics, Yakov Stepanovich Chaikovsky, a very young man at that time. Now, in retrospective of a few decades of my own