LECTURES ON MODULAR CURVES

PETE L. CLARK

1. Some topology of group actions on spaces

Let G be a group and (X, τ_X) a topological space. An **action** of G on (X, τ) is a group action $\rho : G \times X \to X$ on the underlying set X such that for all $g \in G$, the map $g\bullet : X \to X, \ x \mapsto gx$ is continuous (equivalently, is a homeomorphism).[1]

Note that this definition applies to an "abstract group" G, i.e., G is not itself endowed with a topology. In contrast, if $(G, \cdot, \tau_G)$ is a topological group – set G endowed with a group law $\cdot : G \times G \to G$ and a topology G such that the multiplication map $\cdot$ and the inversion map $x \mapsto x^{-1}$ are both continuous – then by an action of (G, τ_G) on (X, τ_X) we mean a continuous group action $\rho : G \times X \to X$.

Exercise: a) Let G be a topological group. Show that an action of G on X in the second sense implies an action of the abstract group G on X in the first sense. b) Let G be an abstract group, and let $\rho : G \times X \to X$ be an action on the underlying set of a topological space (X, τ). Show that ρ is a group action on the topological space X iff ρ is an action of the topological group $(G, \tau_{\text{discrete}})$ on X.

As for any group action on a set X, we have the **orbit space** $G\backslash X$ and a surjective map $\pi : X \to G\backslash X$. If G acts on a topological space X, it is very natural to give $G\backslash X$ the *quotient topology* for the surjective map π.

There is one nice thing to say about this quotient map in full generality.

Lemma 1. *For any topological group G acting on a topological space X, the map $\pi : X \to X/G$ is open.*

Proof. By definition of the quotient topology, we must show that $U \subset X$ open implies $\pi^{-1}\pi U$ open. But $\pi^{-1}\pi U = GU = \bigcup_{g\in G} gU$. $\square$

Question 1. *Suppose X is a "nice" topological space – e.g. Hausdorff, locally compact, a manifold, a manifold with extra structure, and so forth. What properties of the group G and the action ρ will ensure that the quotient space $G\backslash X$ retains the nice properties of X?*

This is really the key question for us, because recall our first main goal in the course: to show that for any congruence subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$, the quotient space $Y(\Gamma) = \Gamma\backslash\mathcal{H}$ can be endowed with the (unique) structure of a $\mathbb{C}$-manifold such that the quotient map $\pi : \mathcal{H} \to Y(\Gamma)$ is a holomorphic map. In particular, as a topological space, we want $Y(\Gamma) = \Gamma\backslash\mathcal{H}$ to have the structure of a real surface.

[1] n practice we never write (X, τ) but instead speak – abusively, but to everyone's taste – of "the topological space X".

At least half of the real work towards this was done by Robert Varley in his second week lectures. Most – but not all – of what remains is of a rather general nature. It is to my taste to present that which can be done in general in suitable generality, so I will take that approach here.

Here is one important class of examples of an action of a topological group on a topological space: let G be any topological group, and let H be any subgroup of G. Then, the restriction of the group law to H endows it with the structure of a topological group, and we may certainly consider the action of H on G given by $\rho : H \times G \to G,\ (h, x) \mapsto hx$. What can we say about the coset space $H\backslash G$?

Lemma 2. *Let H be a subgroup of the topological group G, and consider the coset space G/H, endowed with the quotient topology.*
a) The space G/H is Hausdorff iff H is closed.
b) The space G/H is discrete iff H is open.
c) If H is normal in G, then G/H is a topological group.

Exercise: Prove it.

Now there are places in mathematics where non-Hausdorff spaces come up naturally, but we are not currently in one of those places: when working with the kind of topological spaces that come up in differential geometry and its relatives, non-Hausdorffness is a pathology to be avoided.

As a first step towards this let us agree to consider only **Hausdorff topological groups**. This is not a serious restriction, in view of the following result.

Lemma 3. *Let G be a topological group with identity element $\{e\}$, and let $H = \overline{\{e\}}$. Then H is a closed normal subgroup of G, so G/H is a Hausdorff group. Moreoover the quotient map $G \to G/H$ is universal for homomorphisms of G into a Hausdorff topological group.*

So given a non-Hausdorff topological group we can in a canonical way "smush it down" to a Hausdorff group. The second point is that we probably do not want to mess with nonclosed subgroups because taking quotients by them is especially liabl to lead to non-Hausdorff spaces. But maybe it is not so clear when a subgroup will be closed? In fact for us it will be:

To be sure, I follow Bourbaki: a quasi-compact space is a topological space such that every open cover admits a finite subcover. A compact space is a space which is quasi-compact and Hausdorff. A locally compact space is a Hausdorff space in which each point admits at least one compact neighborhood; equivalently (for Hausdorff spaces!), at every point P there is a local base of compact neighborhods.

Proposition 4. *Let G be a Hausdorff topological group and H a locally compact subgroup. Then H is closed in G. In particular, every discrete subgroup of a Hausdorff group is closed.*

Proof. Let K be a compact neighborhood of the identity in H. Let U be an open neighborhhod of the identity in G such that $U \cap H \subset K$. Let $x \in \overline{H}$. Then there is a neighborhood V of x such that $V^{-1}V \subset U$, so then

$$(V \cap H)^{-1}(V \cap H) \subset K.$$

Since $x \in \overline{H}$, there exists $y \in V \cap H$, and then $V \cap H \subset yK$. Since for every neighborhood W of x, $W \cap V$ is also a neighborhood of x and thus $W \cap V \cap H \neq \emptyset$, $x \in \overline{V \cap H}$. Since yK is compact in the Hausdorff space H, it is closed and thus $x \in \overline{V \cap H} \subset \overline{yK} = yK \subset H$. So H is closed. $\square$

A very favorable class of examples comes from differential geometry.

First, let us agree that a **topological manifold** is a second-countable Hausdorff space X such that each point x admits an open neighborhood U which is homeomorphic to $\mathbb{R}^n$ (the n is allowed to depend on x, although being a continuous function into a discrete space it is constant on connected components). A **Lie group** is a topological group with underlying space a topological manifold.

Let G be a connected Lie group, and let H be a closed subgroup. Then H is a Lie subgroup (Cartan's Theorem), G/H is a topological (in fact, real analytic) manifold and the quotient map $\pi : G \to G/H$ is an H-bundle.

That G/H is a manifold is, I believe, a deep theorem in differential geometry. There is however a sort of converse which is much easier to prove: if a Lie group G acts transitively on a manifold X with compact stabilizers, then for $x \in X$, the natural map $G/\operatorname{Stab}_G(x) \to X$ is a homeomorphism. This enables one to identify many quotients G/H as manifolds simply by finding the correct action of G in nature. For instance:

Example: The special orthogonal group $SO(n)$ acts transitively on the unit sphere S^{n-1} in $\mathbb{R}^n$, and the stabilizer of the north pole may be identified with $SO(n-1)$. It follows that $SO(n)/SO(n-1) \cong S^{n-1}$.

Example: SU(2) acts transitively on S^2; one of the point stabilizers is $U(1)$. Topologically we have $\mathrm{SU}(2) \cong S^3$ and $U(1) \cong S^1$, so we realize S^3 as an S^1-bundle over S^2 – schematically

$$S^1 \to S^3 \to S^1,$$

the **Hopf map**. Quaternionic and octonionic analogues give bundle maps

$$S^3 \to S^7 \to S^4,$$

$$S^7 \to S^{15} \to S^8.$$

In fact these – together with the less interesting

$$S^0 \to S^1 \to S^1$$

given $z \mapsto z^2$ on S^1 – are the only fiber bundles with spheres as the base, total space and fiber: i.e., all such things come from quotients of Lie groups by subgroups!

We will now justify the above claim that the natural map $G/\operatorname{Stab}_G(x) \to X$ is a homeomorphism by appealing to the following result.

First a quick general topology refresher: a topological space is **Lindelöf** if every open cover admits a countable subcover. This is obviously a significant weakening of compactness. Further, any σ-compact space is Lindelöf, and among metrizable spaces, Lindelöf, separable and metrizable are all equivalent.

A **Baire space** is a topological space such that the union of any countably infinite family of closed subsets with empty interior itself has empty interior. The **Baire Category Theorem** asserts that if a space X is completely metrizable (i.e., there is a complete metric inducing the topology on X) or locally compact, then X is a Baire space.

Theorem 5. *Let G be a locally compact, Lindelöf topological group – e.g. a Lie group! – which acts transitively on a locally compact space X, and suppose that for $x \in X$, $H = \mathrm{Stab}_G(x)$ is locally compact. Then the map*

$$\Psi : G/H \to X, \ g \mapsto gx$$

is a homeomorphism.

Proof. Of course Ψ is a bijection: this is the orbit-stabilizer theorem. Let $\pi : G \to G/H$ be the orbit map. For any subset $Y \subset X$,

$$\Psi^{-1}(Y) = \pi(\{g \in G \mid gx \in Y\});$$

since π is an open map, it follows that if Y is open, so is $\Psi^{-1}(Y)$: Ψ is continuous.[2] The matter of the proof is to show that Ψ is an open map: for this, it suffices to take U open in G, $g \in U$ and show that gx is an interior point of Ux. Let V be a compact neighborhood of the identity of G such that $V = V^{-1}$ and $gV^2 \subset U$. If Vx contains an interior point vx, then $gx = gv^{-1}vx$ is an interior point of Ux. Because G is Lindelöf and $\{gV^\circ\}_{g \in G}$ is an open covering of G, we may write $G = \bigcup_{n=1}^\infty g_n V$ for a sequence of elements $g_n \in G$. Then $X = \bigcup_{n=1}^\infty g_n Vx$. We have written the locally compact space X as a countable union of closed subspaces, so by the Baire Category Theorem at least one of these sets must have nonempty interior and thus Vx has nonempty interior. □

So this is an example where quotients work out very nicely. It is not always this way! To show you that we unfortunately do have to be somewhat careful, let me show you a (surprising, to me at least) example of a non-Hausdorff quotient space.

Example: Let $X = \mathbb{R}^2 \setminus \{0\}$ with the usual topology. Let $\Gamma = \mathbb{Z}$ endowed with the discrete topology, acting on X by $n \cdot (x, y) = (2^n x, 2^{-n} y)$. Then the quotient space $\Gamma \backslash X$ is connected and locally homeomorphic to $\mathbb{R}^2$...but is not Hausdorff! Check for yourself that the images of $(1, 0)$ and $(0, 1)$ in the quotient do not admit disjoint open neighborhoods.

This example is disturbing, because in fact the group action has some favorable properties – just not exactly the right ones.

We say a group action G on a space X is **wandering** if for every $x \in X$ there exists an open neighborhood U of X such that $\{g \in G \mid gU \cap U \neq \emptyset\}$ is finite.

Theorem 6. *For a group G acting on a Hausdorff space X, TFAE:*
(i) The action is free and wandering.
(ii) Every point x admits an open neighborhood U such that for all $1 \neq g \in G$, $gU \cap U = \emptyset$.
(iii) $\pi : X \to G\backslash X$ is a Galois covering map.

[2] Note that we have not yet used any of our restrictive hypotheses: in general, Ψ is a continuous bijection.

When these equivalent conditions hold and X is a topological manifold, $X \to G$ is a locally Euclidean space...which need not be Hausdorff.

Proof. This is standard covering space theory: [M, Thm. 81.5]. (Munkres requires X to be connected and locally path connected, but I don't see where this is used.) $\square$

Thus we need a stronger condition than free and wandering to get a Hausdorff quotient. In fact the right condition is the one that Robert introduced in his lectures.

Recall that a map $f : X \to Y$ of topological spaces is **proper** if the preimage of every compact subset of Y is compact in X.

Proposition 7. *Let H be a compact subgroup of a locally compact group G, and let $X = G/H$. Then the quotient map $\pi : G \to X$ is proper.*

Proof. Let $\{U_i\}_{i\in I}$ be an open covering of X such that each U_i has compact closure, and consider the induced open covering $\{\pi(U_i)\}$ of X. If K is a compact subset of X, then there exists a finite subset $J \subset I$ with $K \subset \bigcup_{i\in J} \pi(U_i)$. Then $\pi^{-1}(K)$ is a closed subset of the compact subspace $\bigcup_{i\in J} \overline{U_i}H$, hence is compact. $\square$

Recall that Robert showed this for $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}_2(\mathbb{R})$.

Theorem 8. *Let H be a compact subgroup of a locally compact group G, let $X = G/J$ and $\pi : G \to X$. Let Γ be a subgroup of G. The following are equivalent:*
(i) Γ is discrete.
(ii) For all compact subsets $K_1, K_2 \subset X$, $\{g \in \Gamma \mid gK_1 \cap K_2 \neq \emptyset\}$ is finite.

Proof. (i) $\implies$ (ii): Let K_1, K_2 be compact subsets of X, put $C_i = \pi^{-1}(K_i)$, and let $g \in \Gamma$. If $gK_1 \cap K_2 \neq \emptyset$ then $gC_1 \cap C_2 \neq \emptyset$, so $g \in \Gamma \cap (C_2C_1^{-1})$. By the previous result C_1 and C_2 are compact hence so is $C_2C_1^{-1}$. If Γ is discrete, then $\Gamma \cap (C_2C_1^{-1})$ is compact and discrete, hence finite.
(ii) $\implies$ (i): Let V be a compact neighborhood of the identity $e \in G$, and put $x = \pi(e)$. Then
$$\Gamma \cap V \subset \{g \in \Gamma \mid gx \in \pi(V)\}.$$
Taking $K_1 = \{x\}$ and $K_2 = \pi(V)$, we get that $\Gamma \cap V$ is finite. Thus Γ is discrete. $\square$

This motivates the following definition: a group action Γ on a locally compact space X is **properly discontinuous** if for all compact subsets $K_1, K_2 \subset X$, $\{g \in \Gamma \mid gK_1 \cap K_2 \neq \emptyset\}$ is finite.

Exercise: Let Γ be a group action on a locally compact space X.
a) Show that the action is properly discontinuous iff for all compact subsets $K \subset X$, $\{g \in \Gamma \mid gK \cap K \neq \emptyset\}$ is finite.
b) Show that the action is properly discontinuous iff the **orbit map** $\Gamma \times X \to X \times X$, $(g, x) \mapsto (x, gx)$ is proper.
c) Show that a properly discontinuous action is wandering.
d) Show that the action of Example X.X is *not* properly discontinuous.

Theorem 9. *Let Γ act properly discontinuously on a locally compact space X.*
a) The quotient space $\Gamma\backslash X$ is Hausdorff.
b) If Γ also acts freely, then $X \to \Gamma\backslash X$ is a Galois covering map.
c) If X is a topological manifold and Γ acts freely and properly discontinuously then

$\Gamma\backslash X$ is a topological manifold. If X has extra local structure, then $\Gamma\backslash X$ canonically inherits this structure.

Proof. As usual, let $\pi : X \to \Gamma\backslash X$ be the quotient map.
a) Choose $y_1 \neq y_2 \in \Gamma\backslash X$, let $x_1, x_2 \in X$ be such that $\pi(x_1) = y_1$, $\pi(x_2) = y_2$. Since π is open, it is enough to find neighborhoods N_1, N_2 of x_1, x_2 such that $\pi(N_1) \cap \pi(N_2) = \emptyset$; or equivalently, for all $g \in G$, $N_1 \cap gN_2 = \emptyset$. By local compactness we may choose disjoint compact neighborhoods K_i of x_i. Since the action is properly discontinuous, the set $\{g \in G \mid gK_1 \cap K_2 \neq \emptyset\}$ is finite: if it is empty, we're done already; if not, let the elements be $g_1, \ldots, g_n$. For each i, note that $g_i x_2 \neq x_1$ (by hypothesis x_1 and x_2 lie in different G-orbits) so we may find disjoint compact neighborhods $C_{1,i}$ of x_1 and $C_{2,i}$ of $g_i x_2$. Then we may take $N_1 = K_1 \cap \bigcap_{i=1}^n C_{1,i}$ and $N_2 = K_2 \cap \bigcap_{i=1}^n g_i^{-1} C_{2,i}$.
b) Since a properly discontinuous action is wandering, by assumption that action is free and wandering, so this follows from Theorem 6.
c) Combining parts a) and b), we get that $\Gamma\backslash X$ is a Hausdorff locally Euclidean topological space, i.e., a topological manifold.[3] □

Corollary 10. *Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a torsionfree discrete subgroup. Then $Y(\Gamma) = \Gamma\backslash\mathcal{H}$ has a unique $\mathbb{C}$-manifold structure such that $\pi : \mathcal{H} \to Y(\Gamma)$ is holomorphic.*

References

[M] J. Munkres, *Topology. Second Edition.*

[S] G. Shimura, *Introduction to the Arithmetic of Automorphic Functions.*

[3]Perhaps your definition of a topological manifold includes hypotheses of second countability and/or paracompactness. If so, check that if X has either of these properties, so does $\Gamma\backslash X$.