4. Categoricity: a condition for completeness

By Löwenheim-Skolem, any theory in a countable language which admits infinite models admits models of every infinite cardinality, and indeed, models of any given cardinality elementarily equivalent to any fixed infinite model. Thus the next step in understanding the relation of elementary equivalence is to consider models of a fixed cardinality. In this regard, the following definition captures the simplest possible state of affairs.

Let $\kappa$ be an infinite cardinal. A theory $T$ is $\kappa$-categorical if there exists a unique (up to isomorphism) model of cardinality $\kappa$.

Categoricity leads to completeness as follows:

**Theorem 1.** (Vaught’s Test) Let $T$ be a satisfiable theory with no finite models which is $\kappa$-categorical for some $\kappa \geq |\mathcal{L}|$. Then $T$ is complete.

**Proof.** Suppose $T$ is not complete, and let $\varphi$ be a sentence such that $T \models \varphi$ and $T \not\models \neg \varphi$. Then the extended theories $T_1 := T \cup \neg \varphi$ and $T_2 := T \cup \varphi$ are both satisfiable. Since they do not admit finite models, they both admit infinite models. By Löwencine-Skolem, each $T_i$ admits a model $X_i$ of cardinality $\kappa$. But $X_1$ and $X_2$ disagree about the truth of $\varphi$, so they are not even elementarily equivalent – let alone isomorphic – contradicting the $\kappa$-categoricity of $T$. \[\square\]

Exercise 4.1: For a theory $T$, let $T_\infty$ be the theory of infinite models of $T$, i.e., $T$ augmented with the infinite family of sentences $\varphi_n$, each $\varphi_n$ expressing that the structure has at least $n$ distinct elements. Prove the following variation of Vaught’s Test: let $T$ be a theory admitting an infinite model which is $\kappa$-categorical for some $\kappa \geq \max(\aleph_0, |\mathcal{L}|)$. Then $T_\infty$ is complete. Immediately after seeing the proof, A. Brunyate pointed out the following strengthening.
**Theorem 2.** (Brunyate’s Test) Let $\mathcal{T}$ be a satisfiable theory without finite models. Suppose that there exists an infinite cardinal $\kappa \geq |\mathcal{L}|$ such that any two models of $\mathcal{T}$ of cardinality $\kappa$ are elementarily equivalent. Then $\mathcal{T}$ is complete.

Exercise 4.2: Prove Brunyate’s Test, and also its analogue for $\mathcal{T}_\infty$ as in Exercise 4.1.

One may ask why we use Vaught’s Test and not Brunyate’s Test since the latter is plainly stronger. Indeed, every complete theory satisfies Brunyate’s Test. The answer, I believe, is that the hypothesis of Brunyate’s Test is model-theoretic in nature, whereas the (stronger!) hypothesis of Vaught’s test belongs to mainstream mathematics. Therefore in certain elementary instances we essentially already know that the hypothesis of Vaught’s test is satisfied and stating it as a theorem is a clue to keep one’s eye open for $\kappa$-categorical theories.

We now give some examples of the successful application of Vaught’s test.

**Proposition 3.** Let $\mathcal{L}$ be the empty language – i.e., the language of naked sets. Let $X$ and $Y$ be $\mathcal{L}$-structures. TFAE:

(i) Either $X$ and $Y$ are both infinite, or $X$ and $Y$ are both finite with $|X| = |Y|$.

(ii) $X \equiv Y$.

Exercise 4.3: Prove Proposition 3.

Exercise 4.4: Let $\mathcal{L}$ be the language with a single constant symbol, so $\mathcal{L}$-structures are pointed sets. Classify $\mathcal{L}$-structures up to elementary equivalence.

Exercise 4.5: Let $\mathcal{L}$ be the language with a single unary relation, so $\mathcal{L}$-structures are pairs $(X, Y)$ with $Y \subset X$. Try to classify $\mathcal{L}$-structures up to elementary equivalence.

**Theorem 4.** The theories $\text{ACF}_0$ and $\text{ACF}_p$ (for any prime $p \geq 0$) are each $\kappa$-categorical for any uncountable cardinal $\kappa$. None of these theories admit finite models, so by Vaught’s test they are all complete.

**Proof.** In other words, we claim that if $K_1$ and $K_2$ are algebraically closed fields of the same characteristic and the same uncountable cardinality, then they are isomorphic. This is a true fact of field theory, a consequence of the following more precise result: two algebraically closed fields are isomorphic iff they have the same characteristic and the same absolute transcendence degree (i.e., the transcendence degree over their prime subfield). But for an uncountable field, the transcendence degree is equal to the cardinality. \[\square\]

Remark: The theories $\text{ACF}_0$ and $\text{ACF}_p$ are not $\aleph_0$-categorical: for countable fields, the absolute transcendence degree is an extra invariant. This provides our first example of two structures of the same cardinality which are elementarily equivalent but not isomorphic: say $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}(\ell)}$.

At a deeper level, algebraic geometers have long known that – Lefschetz principle notwithstanding! – a countable algebraically closed field of larger transcendence degree is a “richer” object than $\overline{\mathbb{Q}}$. For instance, not every complex algebraic variety may be defined over $\overline{\mathbb{Q}}$. Among countable models of $\text{ACF}_0$, the “richest”– indeed, the maximal one with respect to embeddings – is clearly the one of countably infinite transcendence degree. Such fields played a fundamental role in Weil’s formalization of algebraic geometry via *universal domains*. Although
this notion is now somewhere between out of fashion and completely forgotten by contemporary algebraic geometers, it is well appreciated by model theorists, being an instance of the notion of a saturated model (which I think we will not get to in this course).

4.1. DLO.

Recall that DLO (dense linear orders) is the theory (well-defined up to syntactic closure) in the language $\mathcal{L} = \{<\}$ consisting of one binary relation whose models are precisely the nonempty linearly ordered sets without endpoints and for which the order relation is dense: for all $x < y$, there exists $z$ with $x < z < y$.

Exercise 4.6: a) Show that DLO does not admit finite models.

b) Let $(F, <)$ be an ordered field. Show that the underlying ordered set is a DLO.

Theorem 5. The theory DLO of dense linear orders without endpoints is $\aleph_0$-categorical. Thus - by Exercise 4.6a) and Vaught’s Test, DLO is complete.

Proof. The proof is by what is called a back and forth argument. Let $X = \{x_n\}_{n=1}^{\infty}$ and $Y = \{y_n\}_{n=1}^{\infty}$ be countable DLOs. We will build up an order-preserving bijection from $X$ to $Y$ via a sequence of countable steps. At Step $2n - 1$, we will ensure that $x_n$ is in the domain of the bijection, and at Step $2n$, we will ensure that $y_n$ is in the codomain of the bijection. If we can do this, we’re done!

Step 1: Take $x_1$ and map it to any element of $Y$.

Step 2: If $y_1$ is already in the codomain of $f_1$, we do nothing. If not, we choose an element $x$ of $X$ and map $x$ to $y_1$. We choose $x$ such that $x < x_1$ if $y_1 < f(x_1)$ and $x > x_1$ if $y_1 > f(x_1)$.

Step 3: If $x_2$ is already in the domain of $f_2$, we do nothing. If not, we choose an element $y$ of $Y$ and map $x_2$ to $y$. We do this in such a way to preserve the extant order relations: the elements in the order-preserving bijection $f_2$ split up both $X$ and $Y$ into finitely many intervals, each of which is nonempty by definition of DLO. So we need only choose $y$ lying in the corresponding interval to $x_2$.

We continue in this manner. A little thought shows that this strategy succeeds. □

Exercise 4.6: a) Show that DLO is not $2^{\aleph_0}$-categorical. (Suggestion: compare $\mathbb{R}$ with its canonical ordering to the ordered sum $\mathbb{R} + \mathbb{Q}$: i.e., we place a copy of $\mathbb{Q}$ “on top of” $\mathbb{R}$ such that every element of $\mathbb{Q}$ is greater than every element of $\mathbb{R}$.)

b)(harder) Show that DLO is not $\kappa$-categorical for any uncountable $\kappa$.

Exercise 4.7: Let DLOE be the theory of dense linear orders with largest and smallest elements.

a) Show that DLOE is $\aleph_0$-categorical. (Hint: use either the statement or the proof of Theorem 5.)

b) Apply Vaught’s Test to show that DLOE is a complete theory.

c) Show that DLOE is not model complete.\(^1\)

The method of proof of Theorem 5 is probably more important than the result

\(^1\)This is a standard example of a complete but not model complete theory. In fact, it is the simplest one I know, although it requires some machinery to show this. If you can think of a more elementary example of a complete but not model complete theory, please let me know!
itself. The construction of an isomorphism, or elementary embedding, by **back and forth** turns out to be one of the most fundamental notions in model theory. Indeed, in Bruno Poizat’s (somewhat idiosyncratic, but extremely insightful) introductory text [Poi], the concept of back-and-forth is taken as a primitive and elementary equivalence is defined in terms of it.

### 4.2. $R$-modules.

In this section we provide a glimpse of the model-theoretic study of modules over a (not necessarily commutative) ring. While perhaps not as sexy as the model theory of fields, this is nevertheless an active research area at the border of model theory and algebra.

As motivation, we provide two examples of complete theories in the language of groups. First some (standard) terminology.

Let $G$ be a group. The **exponent** $E(G)$ is the least positive integer $E$ such that for all $g \in G$, $g^E = e$ – equivalently, the least common multiple of all orders of elements of $G$. (If no such integer exists, we say that the exponent is $\infty$.) For instance, if $G$ is finite, then by Lagrange’s Theorem $E(G) | |G|$. Moreover, for every $n \in \mathbb{Z}^+$, we have a map $[n] : G \to G, g \mapsto g^n$. (Note that $[n]$ need not be a group homomorphism. Indeed $[2]$ is a homomorphism iff $G$ is commutative iff $[n]$ is a homomorphism for all $n \in \mathbb{Z}^+$.) We say that $G$ is **torsionfree** if for all $n \in \mathbb{Z}^+$ and all $g \in G$, $[n]g = e \iff g = e$. $G$ is **divisible** if each $[n]$ is surjective and **uniquely divisible** if each $[n]$ is bijective.

**Theorem 6.** Let $\mathcal{L} = \{+, -, 0\}$ be the language of commutative monoids. All of the following $\mathcal{L}$-theories are complete:

(i) For any prime $p$, the theory of infinite commutative groups of exponent $p$.

(ii) The theory of nontrivial uniquely divisible abelian groups.

**Proof.** Each of the theories admits only infinite models, so it enough to show that these theories are $\kappa$-categorical for some infinite cardinal and then apply Vaught’s test.

A commutative group of exponent $p$ has, in a unique way, the structure of an $\mathbb{F}_p$-vector space, and conversely the additive group of any nontrivial $\mathbb{F}_p$-vector space is a commutative group of exponent $p$. The only invariant of an $\mathbb{F}_p$-vector space is its dimension, and for any infinite $\mathbb{F}_p$-vector space $V$, its dimension is simply equal to its cardinality (c.f. Lemma 7). Therefore the theory of infinite commutative groups of exponent $p$ is $\kappa$-categorical for all infinite $\kappa$.

Similarly, a uniquely divisible abelian group has, in a unique way, the structure of a $\mathbb{Q}$-vector space, and conversely the additive group of any $\mathbb{Q}$-vector space is a commutative, uniquely divisible abelian group. The only invariant of a $\mathbb{Q}$-vector space is its dimension, and for any uncountable $\mathbb{Q}$-vector space $V$, its dimension is simply equal to its cardinality (c.f. Lemma 7). Therefore the theory of nontrivial uniquely divisible commutative groups is $\kappa$-categorical for all uncountble $\kappa$. □

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\(^2\)If $G$ is commutative, this is equivalent to saying that each $[n]$ is injective. For noncommutative $G$, the latter condition is *a priori* stronger, but I don’t have an example to confirm that it is strictly stronger.
The following result nails down the relation between the dimension of a vector space and its cardinality, special cases of which were used in the above proof.

**Lemma 7.** Let $F$ be a field and $V$ a nontrivial vector space over $F$. Then

$$|V| = \max(|F|, \dim(V)).$$

Exercise 4.8: Prove Lemma 7.

These examples suggest a common generalization in terms of vector spaces over a field. However, it is somewhat “lucky” that vector spaces over $\mathbb{F}_p$ and over $\mathbb{Q}$ are characterized by their underlying abelian groups. This cannot be the case in general: e.g. the underlying abelian groups of a $\mathbb{Q}(\sqrt{2})$-vector space are the same as those of a $\mathbb{Q}(\sqrt{3})$-vector space. This suggests a linguistic adjustment: to capture the structure of a vector space over a field $F$, we include the action of the elements of $F$ as part of the language. Indeed, this can be done more generally.

Let $R$ be a ring (not necessarily commutative, but with multiplicative identity). To avoid trivialities, we exclude the zero ring. The language of (say, left) $R$-modules is, by definition, $L_R = \{+, -, 0\} \cup \{r : r \in R\}$, i.e., the language of commutative monoids augmented by a unary function $r$ for each $r \in R$. The class of left $R$-modules and $R$-module homomorphisms is easily seen to be an elementary class of $L_R$-structures: in other words, the usual axioms for a left $R$-module are expressable as sentences in $L_R$.

The theory of $R$-modules is not complete, because the trivial $R$-module has one element, whereas the $R$-module $R$ itself has more than one element. However, there is a class of rings – containing $R = \mathbb{Q}$ as above – such that the theory of nontrivial $R$-modules is complete.

**Theorem 8.** Let $R$ be a ring (not the zero ring!) without zero divisors.

a) The theory of infinite $R$-modules is complete iff $R$ is a division ring.

b) The theory of nontrivial $R$-modules is complete iff $R$ is an infinite division ring.

The following exercises lead a reader through a proof of Theorem 8.

Exercise 4.9: Let $R$ be a ring (not the zero ring!) without zero divisors. An element $x$ in a left $R$-module $M$ is said to be torsion if there exists $0 \neq r \in R$ such that $rx = 0$. A left $R$-module is torsionfree if the only torsion element is zero.

a) Show that $R$ itself is a torsionfree left $R$-module. (We use here that $R$ has no zero divisors.)

b) Let $M$ and $N$ be left $R$-modules. If $M$ is torsionfree and $M \equiv N$ in the language of $R$-modules, then $N$ is torsionfree.

c) Suppose that $R$ is a ring which admits nontrivial torsion left $R$-modules. Show that the theories of nontrivial and infinite left $R$-modules are not complete.

Exercise 4.10: For a ring $R$ (not the zero ring) without zero divisors, show that the following are equivalent:

(i) The only left ideals of $R$ are $\{0\}$ and $R$. (ii) $R$ is a division ring.

(iii) Every left $R$-module is torsionfree.

(iv) Every left $R$-module is isomorphic to the direct sum of $\kappa$ copies of $R$ for a
uniquely determined cardinal $\kappa$.

Exercise 4.11: Prove Theorem 8.

Exercise 4.12: What about the case of rings with zero divisors?\(^3\)

4.3. Morley’s Categoricity Theorem.

One cannot help but notice the dichotomy between countable and uncountable cardinals in all of our applications of Vaught’s test. It is natural to wonder whether there is a theory which is $\kappa$-categorical for some but not all uncountable cardinals. The answer is a resounding no.

**Theorem 9.** (Morley’s Categoricity Theorem) If a theory is categorical for some uncountable cardinal, then it is categorical for every uncountable cardinal.

As Marker remarks in his book [Mar], “Morley’s proof was the beginning of modern model theory.” This theorem is too rich for our blood: beautiful and impressive as it is, it is a theorem of pure model theory: it is hard to imagine a mainstream mathematical problem in which distinct uncountable cardinals arise naturally.\(^4\)

4.4. Complete, non-categorical theories.

It is important to emphasize that Vaught’s test is only a sufficient condition for completeness. (Indeed, it is the “cheapest” criterion for completeness that I know, but as we have seen, it nevertheless has some useful consequences.) There are complete theories which are far from being $\kappa$-categorical for any infinite cardinal $\kappa$. As usual, the theory RCF of real-closed fields (again, either in the language of fields or in that of ordered fields; it doesn’t matter) is an important example.

Exercise 4.13: Let $T$ be a theory in a countable language. Show that $T$ has at most $c = 2^{\aleph_0}$ pairwise nonisomorphic countable models.

Exercise 4.14: Show that RCF, the theory of real-closed fields, has $c$-many pairwise nonisomorphic countable models. Suggestions:

(i) In fact, there are $c$-many countable Archimedean real-closed fields. To see this:

(ii) Show that every real number $\alpha$, there is a countable real-closed subfield $R$ of $\mathbb{R}$ such that $\alpha \in R$.

(iii) Show that any two distinct real-closed subfields of $\mathbb{R}$ are nonisomorphic. (Hint: in a previous exercise you were asked to show that every Archimedean ordered fields order embeds into $\mathbb{R}$. Here we want the fact that this embedding is unique, which is in fact easier to see.)

Exercise 4.15: Let $C$ be an algebraically closed field of characteristic 0. The point of this exercise is to use real-closed fields to show that the automorphism group

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\(^3\)This section is taken from my notes on model theory from 2003, in which I had overlooked the point that “torsionfree” is not a good notion for rings with zero divisors. To make things easy for myself here, I have simply added that hypothesis throughout, but it seems likely that something can be said in the general case.

\(^4\)To those readers who are offended by this statement, my apologies: in Athens, GA there are no practitioners of set theory, topos theory, general topology...
G = Aut(C) is really big.
a) Show that there exists at least one subfield \( R \) of \( C \) such that \( [C : R] = 2 \). (By the Grand Artin-Schreier Theorem, \( R \) is real-closed.) Fix one such subfield and call it \( R_0 \).
b) Let \( H = \{ \sigma \in \text{Aut}(C) \mid \sigma(R_0) = R_0 \} \). Note that \( H \) contains \( h = \text{Aut}(C/R_0) = \{1, c_R\} \), a group of order 2. If the unique ordering on \( R_0 \) is Archimedean, show that \( H = h \).
c) Show that there exist non-Archimedean real-closed fields with \( H \supseteq h \) and also non-Archimedean real-closed fields with \( H = h \). (This is quite difficult.)
d) Show that the coset space \( G/H \) is naturally in bijection with the set of all index 2 subfields \( R \) of \( C \) such that \( R \cong R_0 \).
e) Show that every real-closed field \( R \) with \( |R| = c \) embeds as an index 2 subfield of \( C \).
f) Apply part e) and the previous exercise to show that there are precisely \( 2^c = 2^{2^{2^0}} \) conjugacy classes of order 2 elements in \( G \). In particular \( |G| = 2^{2^{2^0}} \).

References