

HANDOUT TWO: KEPLER'S LAWS OF PLANETARY MOTION

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Rough: Johannes Kepler made three observations about the behavior of the various planets orbiting the sun. A later generation of scientists, trying to understand gravitational forces, postulated an **inverse square law**: namely that the force of gravitational attraction between two bodies should be inversely proportional to the *square* of the distance between them. This guess was made by many scientists, among them Robert Hooke and Isaac Newton. Newton went much further however: he created a new subject of mathematics able to express precisely the relationships between arbitrary changing quantities – “the calculus” – and he used calculus to *derive* Kepler’s laws as mathematical consequences of a body in motion about the sun subject to the force of an inverse square law. It does not seem to overstate matters to describe this achievement as the foremost example of mathematics successfully applied to the understanding of nature¹

The derivation of Kepler’s laws is rather involved; we will work up to it in a number of steps. Suppose $\mathbf{r} = \mathbf{r}(t)$ are the parametric equations of a particle moving in three-dimensional space, subject at any point (x, y, z) to a force $\mathbf{F}(x, y, z)$. Then Newton’s second law reads

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{a}(t) = m\mathbf{r}''(t),$$

or, in words, force equals mass times acceleration.² We also have Newton’s law of universal gravitation:

$$\mathbf{F} = -\frac{GMm}{\|\mathbf{r}\|^2} \hat{\mathbf{r}}.$$

This deserves some elaboration: here we have two bodies (idealized as points) with masses M and m respectively, and \mathbf{r} is the vector from the first body to the second body. The magnitude of the force is then directly proportional to the product of the masses and inversely proportional to the square of the distance $\|\mathbf{r}\|$ between them, and the G is just a proportionality constant which is supposed to be equal for all masses. (That G is astoundingly *small* can be seen in the fact that you do not in any way perceive a force of attraction between yourself and the objects around you: you must have mass on the scale of a planet for the gravitational force to come into play, in contrast to electrical forces: the attraction between a single proton and a single electron is palpable.) But note that Newton’s law is a vector equation: more than just describing the magnitude of the gravitational force, it gives the direction: the force felt by the mass m is opposite in direction to the vector \mathbf{r} from M to m . (By Newton’s third law, there is a force equal in magnitude and opposite in

¹If you have another example in mind, I would like to know!

²It has long seemed to me that one can regard Newton’s second law as the *definition* of force and that it does not actually express any law of nature. If you think otherwise, you can try to convince me by coming up with an example of a conceivable physical situation which would violate $F = ma$.

direction that m exerts on M : in our applications we assume that the mass M is so massive that this force does not accelerate M much, i.e., that the larger mass M is fixed. In reality this is not quite true, but it is an excellent simplifying assumption.)

Combining these two equations we get the equation which we will actually use:

$$(1) \quad \mathbf{a}(t) = \mathbf{r}''(t) = \frac{-Gm}{\|\mathbf{r}\|^2} \hat{\mathbf{r}}.$$

Here then are Kepler's laws:

Kepler's First Law: The orbit of a planet about the sun is elliptical with the sun at one of the foci of the ellipse. (In particular each planet's orbit lies in a single plane.)

Kepler's Second Law: Equal areas are swept out in equal times.

Kepler's Third Law: The square of the period is proportional to the cube of the length of the orbit (i.e., of the length of the semimajor axis of the ellipse). Moreover this proportionality constant is the same for all planets, depending only on the mass of the sun and the universal gravitational constant.

We will need the following strange little fact in the course of the derivation: for any two vectors \mathbf{v} and \mathbf{w} in \mathbb{R}^3 ,

$$(2) \quad (\mathbf{w} \times (\mathbf{v} \times \mathbf{w})) \cdot \mathbf{v} = \|\mathbf{v} \times \mathbf{w}\|^2.$$

To see this, let α be the angle between $\mathbf{w} \times (\mathbf{v} \times \mathbf{w})$ and \mathbf{v} . Then by the dot product formula the left hand side of the expression is equal to

$$\|\mathbf{w} \times (\mathbf{v} \times \mathbf{w})\| \|\mathbf{v}\| \cos \alpha.$$

Similarly, we can use the cross product formula to compute the length of $\mathbf{w} \times (\mathbf{v} \times \mathbf{w})$: note that the second vector is the cross product of \mathbf{v} and \mathbf{w} so it is perpendicular to \mathbf{w} , so the sine of the angle between them is 1 and

$$\|\mathbf{w} \times (\mathbf{v} \times \mathbf{w})\| = \|\mathbf{w}\| \|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{w}\| \|\mathbf{v}\| \|\mathbf{w}\| \sin \theta,$$

where θ is the angle between \mathbf{v} and \mathbf{w} . Substituting this last equation in, the entire left hand side is equal to $\|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \sin \theta \cos \alpha$. But I claim that θ and α are complementary angles, so that $\cos \theta = \sin \alpha$. Let's see why: $\mathbf{v} \times \mathbf{w}$ is the normal vector to the plane generated by \mathbf{v} and \mathbf{w} , and the vector $\mathbf{w} \times (\mathbf{v} \times \mathbf{w})$ is perpendicular to this normal vector, so $\mathbf{u} := \mathbf{w} \times (\mathbf{v} \times \mathbf{w})$ lies in the same plane as \mathbf{v} and \mathbf{w} . But also this vector \mathbf{u} is perpendicular to \mathbf{w} , so the angle from \mathbf{u} to \mathbf{v} – namely α when added to the angle from \mathbf{v} to \mathbf{w} – namely θ – gives the angle from \mathbf{u} to \mathbf{w} , a right angle. So indeed θ and α are complementary angles, so that $\cos \alpha = \sin \theta$ and the left hand side becomes

$$\|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \sin^2 \theta = \|\mathbf{v} \times \mathbf{w}\|^2.$$

Now we begin our derivation of the Kepler's first law. The first order of business is to see that the motion lies in a single plane. Indeed, this plane will be the plane containing the initial velocity vector $\mathbf{v}(0)$ and the initial acceleration vector $\mathbf{r}''(0)$. That the motion cannot escape this plane seems intuitively clear since it starts in the plane and the direction of the force is always back towards the sun, which lies in the plane: at no point do we exert a force which has any component in the

perpendicular direction. We need the expression for the normal vector the plane of motion anyway, so we start with:

Step 1: $\mathbf{r} \times \mathbf{v} = \mathbf{c}$ is a constant, so that \mathbf{c} gives the normal vector for the plane of motion (the fact that $\mathbf{c} \perp \mathbf{r}(t)$ for all t means the motion is confined to a single plane).

Proof: To show that a function is constant, we can show its derivative is zero: $d/dt(\mathbf{r} \times \mathbf{v}) = d\mathbf{r}/dt \times \mathbf{v} + \mathbf{r} \times d\mathbf{v}/dt = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{0}$, since in each case we are taking the cross product of two vectors which lie along the same line: we use here the fact that the acceleration vector lies along the same line as the position vector as given in Equation 1.³

Let $\hat{\mathbf{r}}$ be a unit vector in the direction of \mathbf{r} and write $r = \|\mathbf{r}(t)\|$, so we have $\mathbf{r} = r\hat{\mathbf{r}}$.

Step 2: We calculate

$$\begin{aligned} \mathbf{c} &= \mathbf{r} \times \mathbf{v} = (r\hat{\mathbf{r}}) \times (d\mathbf{r}/dt) = \\ &(r\hat{\mathbf{r}} \times (rd\hat{\mathbf{r}}/dt + dr/dt\hat{\mathbf{r}})) = r^2(\hat{\mathbf{r}} \times d\hat{\mathbf{r}}/dt) + r dr/dt(\hat{\mathbf{r}} \times \hat{\mathbf{r}}) = r^2(\hat{\mathbf{r}} \times d\hat{\mathbf{r}}/dt). \end{aligned}$$

Step 3: We claim that $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = 0$. Indeed, since $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \|\hat{\mathbf{r}}\|^2 = 1$, we have

$$0 = d/dt(1) = d/dt(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) = \hat{\mathbf{r}}' \cdot \hat{\mathbf{r}} + \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = 2(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}').$$

Step 4:

$$d/dt(\mathbf{v} \times \mathbf{c}) = \mathbf{r}'' \times \mathbf{c} + \mathbf{v} \times d\mathbf{c}/dt \stackrel{*}{=} \mathbf{r}''(t) \times \mathbf{c} = (-GM/r^2\hat{\mathbf{r}}) \times (r^2(\hat{\mathbf{r}} \times \hat{\mathbf{r}}')) = -GM(\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{r}}')).$$

Now by Step 3 $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ are perpendicular, so in a suitable coordinate system $\hat{\mathbf{r}}, \hat{\mathbf{r}}' \times \hat{\mathbf{r}}$ can be thought of as $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. Since $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$, we have also that

$$\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{r}}') = -\hat{\mathbf{r}}'$$

. Thus:

$$d/dt(\mathbf{v} \times \mathbf{c}) = GM\hat{\mathbf{r}}'.$$

Integrating this equation, we get

$$(3) \quad \mathbf{v} \times \mathbf{c} = GM\hat{\mathbf{r}} + \mathbf{d},$$

where \mathbf{d} is a constant vector.

Step 5: Dot both sides of Equation 3 with \mathbf{r} , getting

$$(\mathbf{v} \times \mathbf{c}) \cdot \mathbf{r} = (\mathbf{v} \times (\mathbf{r} \times \mathbf{v})) \cdot \mathbf{r} = (GM\hat{\mathbf{r}} + \mathbf{d}) \cdot \mathbf{r}.$$

We need to simplify both sides: as for the left hand side, it is the reason we proved the identity of Equation 2, applying it, the left hand side is $\|\mathbf{r} \times \mathbf{v}\|^2 = \|\mathbf{c}\|^2 = c^2$, say. As for the right hand side, let θ be the angle between $\hat{\mathbf{r}}$ and \mathbf{d} , and we get

$$(GM\hat{\mathbf{r}} + \mathbf{d}) \cdot \mathbf{r} = GMr + dr \cos \theta = rGM(1 + \frac{d}{GM} \cos \theta).$$

³Note indeed that we do not need the inverse square law for this step: assuming only that the sun pulls back on the planet along the vector connecting them, it follows that the motion will lie in a single plane.

Equating the left and right sides, we get

$$(4) \quad r = \frac{\frac{c^2}{GM}}{1 + \frac{d}{GM} \cos \theta} = \frac{p}{1 + e \cos \theta},$$

where we have taken $p := \frac{c^2}{GM}$, $e := \frac{d}{GM}$. We are now essentially done: we just have to put this polar equation in rectangular coordinates. Clearing the denominator, we get

$$r + er \cos \theta = p,$$

or

$$r^2 - (p - er \cos \theta)^2.$$

Now $r^2 = x^2 + y^2$ and $x = r \cos \theta$, so we get

$$\begin{aligned} x^2 + y^2 - (p - ex)^2 &= p^2 - 2epx + e^2x^2. \\ (1 - e^2)x^2 + 2epx + y^2 &= p^2. \end{aligned}$$

After all this work, there is an interesting surprise: we do not *necessarily* get an ellipse: for this we need $1 - e^2 < 0$ or $e < 1$. If $e = 1$ we get a parabola and if $e > 1$ we get a branch of a hyperbola. Both of these are indeed possible as motions of “planets” whose orbits are not periodic, like comets. In any case, we can clean up the equation a bit by dividing by $1 - e^2$ and complete the square with respect to x :

$$\begin{aligned} x^2 + \frac{2ep}{1 - e^2}x + \frac{y^2}{1 - e^2} &= \frac{p^2}{1 - e^2}. \\ \left(x + \frac{ep}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} &= \frac{p^2}{1 - e^2} + \left(\frac{ep}{1 - e^2}\right)^2 = \frac{p^2}{(1 - e^2)^2}. \\ \left(x + \frac{ep}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} &= \frac{p^2}{(1 - e^2)^2}. \\ \frac{\left(x + \frac{ep}{1 - e^2}\right)^2}{p^2/(1 - e^2)^2} + \frac{y^2}{p^2/(1 - e^2)} &= 1. \end{aligned}$$

Assuming $e < 1$, this is an ellipse with semimajor/minor axes $a^2 = p^2/(1 - e^2)^2$ and $b^2 = p^2/(1 - e^2)$. By definition, the focus is c , where $c^2 = a^2 - b^2 = \frac{p^2e^2}{(1 - e^2)^2}$. So indeed the ellipse is centered at $(-\frac{ep}{1 - e^2}, 0) = (-c, 0)$.

Thankfully, the derivation of the other two laws is much easier. Kepler’s second law means the following: consider the line segment connecting the sun to the planet travelling around on its elliptical orbit. This segment sweeps out a certain area, a “sector of the ellipse.” To say that it sweeps out equal areas in equal times means: dA/dt is constant. Before we prove this, it is worth noting that this would certainly not be true for the “standard parameterization” $(a \cos \theta, b \sin \theta)$ of the ellipse $x^2/a^2 + y^2/b^2 = 1$, because this parameterization is symmetrical: the particle is travelling at the same speed when it crosses the x -axis both at the left-most point (nearest the focus) and at the right-most point (farthest away from the focus), but if the areas are to be equal, the planet must move *fastest* along the path when it is closest to the sun and *slowest* when it is farthest away.

Anyway, we compute the area in polar coordinates as

$$A(\theta_2) - A(\theta_1) = \int_{\theta_1}^{\theta_2} 1/2r^2(\varphi)d\varphi.$$

(We will discuss this formula more fully later on in the course.) The fundamental theorem of calculus tells us that if we differentiate an integral we get back the function we started with, so $dA/d\theta = 1/2r^2(\theta)$. But careful: we want the rate of change of area with respect to *time*, not with respect to the angle. The angle is itself changing with respect to time, so we have a chain rule situation:

$$dA/dt = dA/d\theta d\theta/dt = 1/2r^2(\theta)d\theta/dt.$$

To compute $d\theta/dt$, we write $\hat{\mathbf{r}} = (\cos \theta, \sin \theta)$, so that $\hat{\mathbf{r}}' = (-\sin \theta d\theta/dt, \cos \theta d\theta/dt)$. Recall we had $\mathbf{c} = r^2(\hat{\mathbf{r}} \times \hat{\mathbf{r}}')$ for a constant vector \mathbf{c} . Substituting in, we get

$$\mathbf{c} = r^2(d\theta/dt\hat{\mathbf{k}}),$$

so $c = \|\mathbf{c}\| = r^2 d\theta/dt$, so $dA/dt = 1/2r^2(c/r^2) = 1/2c$, a constant as claimed.

Finally, for Kepler's third law, recall that the total area of an ellipse with semimajor axis a and semiminor axis b is πab . (If you don't remember this formula, you can derive it by evaluating the definite integral $\int_{-a}^a \sqrt{b^2(1 - (x/a)^2)} dx$ using the substitution $x = a \sin \theta$.) The **period** T of the orbit is the time it takes for the planet to go once around the sun. So if we integrate the rate of change of area from 0 to T we'll get the total area:

$$\pi ab = \int_0^T dA/dt dt = \int_0^T 1/2c dt = 1/2cT.$$

Thus $T = 2\pi ab/c$ hence $T^2 = 4\pi^2 a^2 b^2 / c^2$. Using

$$b^2 = \frac{p^2}{1 - e^2}, a = \frac{p}{1 - e^2}$$

we get $b^2 = pa$. Also

$$p = c^2/GM$$

so

$$T^2 = \frac{4\pi^2 a^2 (pa)}{pGM} = \left(\frac{4\pi^2}{GM}\right) a^3.$$

Thus T is proportional to $a^{3/2}$, which is Kepler's third law.

This was the most involved calculation we will see in this course. But if you thought it was a bit much, imagine the scientists at the close of the 17th century, completely unfamiliar in the ways of derivatives, integrals, dot and cross products: can you imagine what it must have looked like to them? I think it very instructive to see what the derivation takes, because it makes clear that guessing Newton's inverse square law of gravitation (as Hooke did, among others), it is by no means clear that Kepler's laws must follow.