HANDOUT ONE: MORE ON DOT PRODUCTS AND CROSS PRODUCTS

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Few things are more basic to the study of geometry in two and three dimensions than the dot and cross product of vectors. On the philosophy that it is good to understand simple things very well, I give here a treatment of these two products which is more complete than the one in the textbook (and than the presentation given in class). The emphasis is on an understanding of the following two product formulas:

For any vectors v, w in $\mathbb{R}^2$ or in $\mathbb{R}^3$, if the angle between them is θ, then

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\ ||\mathbf{w}|| \cos\theta. \tag{1}$$

For any vectors v and w in $\mathbb{R}^3$ (only!), if the angle between them is θ then

$$||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||\ ||\mathbf{w}|| \sin\theta. \tag{2}$$

Moreover, the *direction* of the cross-product is determined by the following information: $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, and the triple $(\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w})$ is oriented according to the right-hand rule.

Your textbook takes these equations as the definitions for the dot and cross product of two vectors. I prefer to regard them as properties of the operations, which are defined directly by algebraic formulas.

1. The dot product formula

Let us look first at the dot product: if $\mathbf{v} = (a, b, c)$ and $\mathbf{w} = (x, y, z)$ then $\mathbf{v} \cdot \mathbf{w} = (a, b, c) \cdot (x, y, z) := ax + by + cz$.[1] The dot product of vectors in the plane is defined similarly, just without the last coordinate: $(a, b) \cdot (x, y) := ax + by$. Indeed, the dot product can be defined for pairs of vectors with n components as $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) := x_1y_1 + x_2y_2 + \ldots + x_ny_n$.

Whenever you see a new operation – especially a new kind of "multiplication" – you should not assume that it has all the nice properties that multiplication of ordinary numbers. Notice that the dot product is a rather weird thing: given two vectors, we combine them to get a *scalar*, which may not remind us of anything at all. Here are some properties:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$$

$$\mathbf{u} + (\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$$

$$(c\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (c\mathbf{w}) = c(\mathbf{v} \cdot \mathbf{w})$$

[1] The ":=" is computer science notation meaning that the quantity to the left of the symbol is defined to be equal to the quantity to the right of the symbol.

The last one says that if we scale either one of the vectors in the dot product, then the dot product scales by the same factor. Thus

$$\mathbf{v} \cdot \mathbf{w} = (||\mathbf{v}||\hat{\mathbf{v}}) \cdot (||\mathbf{w}||\hat{\mathbf{w}}) = ||\mathbf{v}|| \, ||\mathbf{w}||\hat{\mathbf{v}} \cdot \hat{\mathbf{w}}.$$

Recall here that the notation $\hat{\mathbf{v}}$ for a unit vector in the direction of $\mathbf{v}$ was suggested (and used!) in Wednesday's class. Thus for every vector $\mathbf{v}$ (except the zero vector) we have $\mathbf{v} = ||\mathbf{v}||\hat{\mathbf{v}}$ expressing $\mathbf{v}$ as a magnitude and a direction. Thus to prove the dot product formula, it suffices to show that the dot product of two *unit* vectors is the cosine of the angle between them.

This is easy for vectors in $\mathbb{R}^2$: to say they are unit vectors just means that they lie on the unit circle. Thus $\hat{\mathbf{v}} = (\cos\alpha, \sin\alpha)$ and $\hat{\mathbf{w}} = (\cos\beta, \sin\beta)$ say, where $\hat{\mathbf{v}}$ make angles of α and β with the x-axis respectively. Note that the angle between them is $\theta = \beta - \alpha$ (draw a picture!). Now we compute

$$\hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = (\cos\alpha, \sin\alpha) \cdot (\cos\beta, \sin\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta =$$
$$\cos(\alpha - \beta) = \cos(\beta - \alpha) = \cos(\theta).$$

In three dimensional space, we had better do something more conceptual lest we get embroiled in spherical geometry. The dot product of two unit vectors is supposed to depend only upon their relative position, so if we rotate both vectors the same amount about any axis, their dot product is supposed to be unchanged. (Even so, the derivation is more involved than I had thought, and uses some ideas on parameterized curves that we will discuss next week.)

Suppose $P = (x, y, z)$ and we rotate the vector OP about the x-axis. (Stop and try to picture this.) The point describes a circle, perpendicular to the x-axis, whose radius is the distance from P to the x-axis, which is $||P - proj_{\hat{i}}P|| = ||(x, y, z) - (x, 0, 0)|| = \sqrt{y^2 + z^2} = R$, say; so the equations describing it are $(x, R\cos\theta, R\sin\theta)$; we get the point P itself when $\tan\theta = z/y$. Now take two vectors $\mathbf{v}$ and $\mathbf{w}$ and represent them in this way:

$$\mathbf{v} = (x_1, R_1\cos\theta_1, R_1\sin\theta_1)$$
$$\mathbf{w} = (x_2, R_2\cos\theta_2, R_2\sin\theta_2)$$

We have

$$\mathbf{v} \cdot \mathbf{w} = x_1x_2 + R_1R_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = x_1x_2 + R_1R_2\cos(\theta_1 - \theta_2).$$

Now rotate both vectors through any common angle α about the x-axis, getting

$$\mathbf{v}_\alpha = (x_1, R_1\cos(\theta_1 + \alpha), R_1\sin(\theta_1 + \alpha)), \ \mathbf{w}_\alpha = (x_2, R_2\cos(\theta_2 + \alpha), R_2\sin(\theta_2 + \alpha)).$$

Now we compute

$$\mathbf{v}_\alpha \cdot \mathbf{w}_\alpha = x_1x_2 + R_1R_2(\cos(\theta_1 + \alpha)\cos(\theta_2 + \alpha) + \sin(\theta_2 + \alpha)\sin(\theta_2 + \alpha)) =$$
$$x_1x_2 + R_1R_2(\cos((\theta_1 + \alpha) - (\theta_2 + \alpha)) =$$
$$x_1x_2 + R_1R_2\cos(\theta_1 - \theta_2) = \mathbf{v} \cdot \mathbf{w}.$$

So indeed, when we rotate a pair of vectors about the x-axis, their dot product doesn't change. By symmetry, the same holds for rotation about the y-axis and the z-axis[2] But now consider that starting with any pair of vectors in $\mathbb{R}^3$, by rotating

[2]In other words, we see that if we wrote the coordinates in a different order, the calculation would be essentially the same.

them about the three coordinate axes we can put both vectors in the xy-plane – since we moved them jointly, the angle between them is unchanged. Finally we can apply the dot product formula in $\mathbb{R}^2$.

An important consequence of the dot product formula (and the derivation!) is that it shows that the dot product of two vectors is **intrinsic**: if we took two wooden sticks of length $||\mathbf{v}||$ and $||\mathbf{w}||$ respectively and attached them to each other with a fixed angle θ between them and pinned the vertex to the origin in $\mathbb{R}^3$, then no matter how we spin this contraption about, the dot product remains the same!

2. The cross product formula

It cannot be overemphasized that the dot product of two vectors is not really a "product" at all in the sense that it given two objects of the same sort (vectors), it returns an object of a different sort (a scalar, or number). It seems more natural to look for vector products of two vectors. Indeed, one might be tempted to look at the following operation:

$$\mathbf{v} \star \mathbf{w} := (a, b, c) \star (x, y, z) = (ax, by, cz)$$

In other words, we try to multiply the vectors the same way we add them: componentwise. This operation is perfectly well-defined; the problem is that, unlike the dot product and the cross product to be seen shortly, it has no use, or more precisely, no geometric or physical interpretation. For instance, the dot product is zero if and only if the vectors are perpendicular to each other, whereas $\mathbf{v} \star \mathbf{w} = (0, 0, 0)$ only if either $a = 0$ or $x = 0$, either $b = 0$ or $y = 0$ and either $c = 0$ or $z = 0$. There's nothing here worthy of our attention.

On the other hand, there is a much crazier looking product, which turns out to be far more useful, namely the **cross product**:

$$\mathbf{v} \times \mathbf{w} = (a, b, c) \times (x, y, z) := (bz - cy, -(az - cx), ay - bx).$$

To remember the formula, we can give it in a more succinct way, using determinants: let $\hat{\imath} = (1, 0, 0)$, $\hat{\jmath} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$ be the unit vectors along the x, y and z axes respectively. Then the cross product of $\mathbf{v}$ and $\mathbf{w}$ can be computed as a 3×3 determinant of the "matrix"[3]

$$\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a & b & c \\ x & y & z \end{bmatrix}$$

or $\hat{\mathbf{i}}(bz - cy) - \hat{\mathbf{j}}(az - cx) + \hat{\mathbf{k}}(ay - bx)$.

The most important property of the cross product of $\mathbf{v}$ and $\mathbf{w}$ is that it is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. Let's check this:

$$(\mathbf{v}\times\mathbf{w})\cdot\mathbf{v} = (bz-cy, cx-az, ay-bx)\cdot(a, b, c) = abz-bcy+bcx-abz+acy-bcx = 0.$$

$$(\mathbf{v}\times\mathbf{w})\cdot\mathbf{w} = (bz-cy, cx-az, ay-bx)\cdot(x, y, z) = bxz-cxy+cxt-ayz+ayz-bxz = 0.$$

[3]The quotation marks are there because some of the entries in the matrix are themselves vectors and others are scalars, which is not the usual state of affairs.

We would also like to know the magnitude of the cross product: recall that we are supposed to have the formula

$$||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||||\mathbf{w}|| \sin\theta.$$

This is a very exciting thing for the magnitude to be: it implies that the cross product is only zero when the angle between $\mathbf{v}$ and $\mathbf{w}$ is zero, i.e., unless $\mathbf{v}$ and $\mathbf{w}$ are scalar multiples of each other. Moreover, the parallelogram formed by adding $\mathbf{v}$ and $\mathbf{w}$ has base length $||\mathbf{w}||$ and height $||\mathbf{v}|| \sin\theta$, so the magnitude of the cross product is precisely the *area* of this parallelogram. Finally, we cannot help but be struck by the similarity to the formula for $|\mathbf{v} \cdot \mathbf{w}|$, which is the same except with $\cos\theta$. This allows us to guess the **Lagrange identity**

$$||\mathbf{v} \times \mathbf{w}||^2 + |\mathbf{v} \cdot \mathbf{w}|^2 = ||\mathbf{v}||^2||\mathbf{w}||^2.$$

Now we are in a sneaky situation: we can apply "Littlewood's Principle" which is (no kidding): purely algebraic identities are very easy to check once someone else has written them down for you. (In other words, geometry and trigonometry can get difficult, but purely algebraic manipulations are always rather routine.) So:

$$||(a,b,c)\times(x,y,z)||^2+|(a,b,c)\cdot(x,y,z)|^2 = (bz-cy)^2+(az-cx)^2+(ay-bx)^2+(ax+by+cz)^2 =$$
$$b^2z^2-2bcyz+c^2y^2+a^2z^2-2acxz+c^2x^2+a^2y^2-2abxy+b^2x^2+a^2x^2+b^2y^2+c^2z^2+2abxy+2acxz+2bcyz =$$
$$(a^2+b^2+c^2)(x^2+y^2+z^2) = ||\mathbf{v}||^2||\mathbf{w}||^2.$$

Using the Lagrange identity and the dot product formula, we can derive the formula for the magnitude of the cross product, namely:

$$||\mathbf{v} \times \mathbf{w}||^2 = ||\mathbf{v}||^2||\mathbf{w}||^2 - |\mathbf{v} \cdot \mathbf{w}|^2 =$$
$$||\mathbf{v}||^2||\mathbf{w}||^2(1-\cos^2\theta) = ||\mathbf{v}||^2||\mathbf{w}||^2 \sin^2\theta$$

and taking square roots we get $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||||\mathbf{w}|| \sin\theta$.

The final bit about the cross product is its orientation: we know it is perpendicular to $\mathbf{v}$ and to $\mathbf{w}$ and has a certain magnitude, so this determines it up to a sign. Why does it come out to be the right-hand rule – and indeed, how could we see such a thing without contorting our poor right hand to every possible angle? It comes down to the simple fact that

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$

as you should check directly, so the right-hand rule holds in this one case. Now, the cross product was given as a simple polynomial function in the entries of the two vectors, so it is a **continuous** function of $\mathbf{v}$ and $\mathbf{w}$: if we change $\mathbf{v}$ and $\mathbf{w}$ just a little bit, so too is the change in $\mathbf{v} \times \mathbf{w}$ small. So as we start with $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ and slowly rotate our hand to put the fingers at $\mathbf{v}$ and the ball of the hand at $\mathbf{w}$, it would be a *big* change if at any given time the cross product suddenly flipped to be in the opposite direction than the right hand rule predicts. In other words, given that the right hand rule holds for some pair of vectors, a failure of it to hold at any other pair would mean a discontinuity in the cross product, which is not possible.