HANDOUT NINE: THE CHANGE OF VARIABLES FORMULA

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The change of variables formula is based on the u-substitution in single variable calculus, or more precisely on the "inverse substitution": say we have $\int_{x_m}^{x_M} g(x)dx$, and we want to make the substitution $x = f(u)$. Then $dx = f'(u)du$ and

$$\int_{x_m}^{x_M} g(x)dx = \int_{u_m}^{u_M} g(f(u))\frac{df}{du}du.$$

Here we must change the x-limits to u-limits. Since $x = f(u)$, $u = f^{-1}(x)$, the inverse function to x. So the substitution gives:

$$\int_{x_m}^{x_M} g(x)dx = \int_{f^{-1}(x_m)}^{f^{-1}(x_M)} g(f(u))(df/du)du.$$

But notice that it would not necessarily be correct to write $u_m = f^{-1}(x_m)$, $u_M = f^{-1}(x_M)$: that is, it may be that after the change of variables the lower limit is actually a larger number than the upper limit. Indeed, this happens exactly when the function $u = f^{-1}(x)$ is *decreasing*: for instance, suppose $[x_m, x_M] = [1, 2]$ and $x = 1/u$, so $u = 1/x$. Then $u(1) = 1$ and $u(2) = \frac{1}{2}$, so the upper and lower endpoints are reversed. But recall that a function is decreasing if and only if its derivative is negative, and that the derivative of f^{-1} is negative if and only if the derivative of f is negative, since $f^{-1\prime}(x) = \frac{1}{f'(f^{-1}(x))}$. Therefore, letting u_m be the smallest u-value – i.e., whichever of $f^{-1}(x_m)$ and $f^{-1}(x_M)$ is smaller – and u_M be the largest u-value, the substitution can also be written as

$$\int_{x_m}^{x_M} g(x)dx = \int_{u_m}^{u_M} g(f(u))|\frac{df}{du}|du,$$

the reason being that if f is increasing, $|\frac{df}{du}| = \frac{df}{du}$ and $u_M = f^{-1}(x_M)$, so the integral is the same as before, whereas if f is decreasing, $|\frac{df}{du}| = -\frac{df}{du}$ but $u_M = f^{-1}(x_m)$, so in switching the upper and lower limits to put u_M on top and replacing $\frac{df}{du}$ with $|\frac{df}{du}|$, we introduce *two* minus signs, which is as good as no minus signs at all.

All this is to explain why the change of variables formula in several variables is a generalization of the u-substitution.

Indeed, supppose we have a double integral $\int\int_R g(x, y)dxdy$ and we want to change to new variables u, v realted to x, y by

$$x = f_1(u, v), \ y = f_2(u, v).$$

We define the **Jacobian** $J\left(\frac{x,y}{u,v}\right)$ to be the two-by-two determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}.$$

Then the change of variables formula reads

$$\int\int_R g(x,y)dxdy = \int\int_{R'} g(f_1(u,v), f_2(u,v))|J\left(\frac{x,y}{u,v}\right)|dudv.$$

Note first that we have taken the absolute value of the Jacobian, and second that the region R' means "R written in the (u,v)-variables": technically, R' is the set of all points (u,v) such that $(f_1(u,v), f_2(u,v))$ is a point of R, but this is an unhelpfully abstract way of thinking about things: we would only change to (u,v)-variables in the first place if R had a nice(r), simple(r) description in terms of the new variables.

It works the same in three variables, namely if

$$x = f_1(u,v,w),\ y = f_2(u,v,w),\ z = f_3(u,v,w),$$

then

$$\int\int\int_V g(x,y,z)dxdydz = \int\int\int_{V'} g(f_1(u,v,w), f_2(u,v,w), f_3(u,v,w))|J\left(\frac{x,y,z}{u,v,w}\right)|dudvdw,$$

where

$$J\left(\frac{x,y,z}{u,v,w}\right) = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}.$$

Example (Polar coordinates): We have

$$x = r\cos\theta = f_1(r,\theta), y = r\sin\theta = f_2(r,\theta).$$

Thus we compute

$$J(\frac{x,y}{r,\theta}) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta = r\sin^2\theta = r,$$

and we've recovered the fact that in polar coordinates $dA = rdrd\theta$.

Example (spherical coordinates): We have

$$x = \rho\cos\theta\sin\varphi,\ y = \rho\sin\varphi\sin\theta, z = \rho\cos\varphi.$$

We will calculate

$$J(\frac{x,y,z}{\rho,\theta,\varphi})$$

$$= \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \sin\theta\cos \\ \cos\varphi & 0 & -\sin\varphi \end{vmatrix} =$$

$$\cos\varphi(-\rho^2\sin^2\theta\sin\varphi\cos\varphi - \rho^2\cos^2\theta\sin\varphi\cos\varphi) - \rho\sin\varphi(\rho\cos^2\theta\sin^2\varphi - \sin\varphi\sin^2\varphi)$$

$$= -\rho^2\sin\varphi.$$

Thus $|J(\frac{x,y,z}{r,\theta,\varphi})| = \rho^2\sin\varphi$, as was claimed earlier in the course.

Note that your textbook gets $\rho^2\sin\varphi$ as a Jacobian rather than with a minus sign. This is correct, since it computes the Jacobian with the variables in a different order: (ρ,φ,θ), rather than our (ρ,θ,φ). (In general, switching two columns of a matrix multiplies the determinant by -1.) Since we can order the new variables however we want, we definitely need to take absolute values when applying the change of variables formula.

Example (Volume of an ellipsoid): Let V be the space region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. We will find the volume of V by changing variables and using the fact that the volume of the region bounded by the unit sphere is $\frac{4}{3}\pi$.

Indeed, consider the change of variables

$$x = au, \ y = bv, \ z = cw.$$

Under this change of variable the equation of the boundary surface becomes $u^2 + v^2 + w^2 = 1$, which is just the unit sphere in (u, v, w)-variables. Therefore

$$\text{vol}(V) = \int\int\int_V 1dV = \int\int\int_{V'} |J(x,y,z)(u,v,w)|dudvdw = |J(\frac{x,y,z}{u,v,w}))|\,\text{vol}(V')$$

$$= \frac{4\pi}{3}|J(\frac{x,y,z}{u,v,w}|.$$

We leave it for you to compute that $J(\frac{x,y,z}{u,v,w}) = abc$, so that the volume of the region V is $\frac{4\pi}{3}abc$.