CONVERGENCE

PETE L. CLARK

Contents

1. Introduction: Convergence Via Sequences and Beyond

Recall the notion of convergence of sequences in metric spaces. In any set X, a sequence in X is just a mapping $\mathbf{x} : \mathbb{Z}^+ \to X$, $n \mapsto \mathbf{x}_n$. If X is endowed with a metric d, a sequence $\mathbf{x}$ in X is said to **converge** to an element x of X if for all $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that for all $n \geq N$, $d(x, x_n) < \epsilon$. We denote this by $\mathbf{x} \to x$ or $\mathbf{x}_n \to x$. Since the ϵ-balls around x form a local base for the metric topology at x, an equivalent statement is the following: for every neighborhood U of x, there exists an $N = N(U)$ such that for all $n \geq N$, $\mathbf{x}_n \in U$.

We have the allied concepts of limit point and subsequence: we say that x is a **limit point** of a sequence $\mathbf{x}_n$ if for every neighborhood U of x, the set of $n \in \mathbb{Z}^+$ such that $\mathbf{x}_n \in U$ is infinite. A **subsequence** of $\mathbf{x}$ is obtained by choosing an infinite subset of $\mathbb{Z}^+$, writing the elements in increasing order as $n_1, n_2, \ldots$ and then

Date: October 18, 2016.

Thanks to Peter Tamaroff – and several others, of whom I seem unfortunately not to have kept track – for pointing out typos.

restricting the sequence to this subset, getting a new sequence $\mathbf{y}$, $k \mapsto \mathbf{y}_k = \mathbf{x}_{n_k}$.

The study of convergent sequences in the Euclidean spaces $\mathbb{R}^n$ is one of the mainstays of any basic analysis course. Many of these facts generalize immediately to the context of an arbitrary metric space (X, d).[1]

Proposition 1.1. *Each sequence in (X, d) converges to at most one point.*

Proposition 1.2. *Let Y be a subset of (X, d). For $x \in X$, TFAE:*
a) $x \in \overline{Y}$.
b) There exists a sequence $\mathbf{x} : \mathbb{Z}^+ \to Y$ such that $\mathbf{x}_n \to x$.

In other words, the closure of a set can be realized as the set of all limits of convegent sequences contained in that set.

Proposition 1.3. *Let $f : X \to Y$ be a mapping between two metric spaces. TFAE:*
a) f is continuous.
b) If $\mathbf{x}_n \to x$ in X, then $f(\mathbf{x}_n) \to f(x)$ in Y.

In other words, continuous functions between metric spaces are characterized as those which preserve limits of convergent sequences.

Proposition 1.4. *Let $\mathbf{x}$ be a sequence in (X, d). For $x \in X$, TFAE:*
a) The point x is a limit point of the sequence $\mathbf{x}$.
b) There exists a subsequence $\mathbf{y}$ of $\mathbf{x}$ converging to x.

Moreover, there are several results in elementary real analysis that exploit, in various ways, the compactness of the unit interval $[0, 1]$:

Theorem 1.5. *(Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Theorem 1.6. *(Heine-Borel) A subset of the Euclidean space $\mathbb{R}^n$ is compact iff it is closed and bounded.*

There are several criteria for compactness in metric spaces. Two of the most important ones are given in the following theorem. Recall that in any topological space X, we say that a point x is a **limit point** of a subset A if for every neighborhood N of x we have $N \setminus \{x\} \cap A \neq \emptyset$. (In other words, x lies in the closure of $A \setminus \{x\}$.)

Theorem 1.7. *Let (X, d) be a metric space. TFAE:*
a) Every sequence has a convergent subsequence.
b) Every infinite subset has a limit point.
c) Every open cover $\{U_i\}$ of X has a finite subcover (i.e., X is compact).

Theorem 1.7 is of a less elementary character than the preceding results, and we shall give a proof of it later on.

These results show that, in a metrizable space, all the important topological notions can be captured in terms of convergent sequences and subsequences. Since every student of mathematics receives careful training on the calculus of convergent sequences, this provides significant help in the topological study of metric spaces.

[1]We recommend that the reader who finds any of these facts unfamiliar should attempt to verify them on the spot. On the other hand, more general results are coming shortly.

It is desirable to have an analogous theory of convergence in arbitrary topological spaces. One can formulate the notion of a convergent sequence in a topological space X, and we will do so. However, we shall see that *none* of the above results hold for sequences in an arbitrary topological space.

There are two reasonable responses to this. First, we can search for sufficient, or necessary and sufficient, conditions on a space X for these results to hold. In fact relatively mild sufficient conditions are not so difficult to find: the Hausdorff axiom ensures the uniqueness of limits; for most of the other properties the key result is the existence of a countable base of neighborhoods at each point.

The other response is to find a suitable replacement for sequences which renders correct all of the above results in an arbitrary topological space. This is a useful enterprise, because there are "in nature" topological spaces which are not Hausdorff (e.g. Zariski topologies in algebraic geometry) or which do not admit a countable neighborhood base at each point (e.g. weak topologies in functional analysis). However, the failure of the above results to hold should suggest to the student of topology that there is "something else out there" which is the correct way to think about convergence in topological spaces. Knowing the "correct" notion of convergence leads to positive results in the theory as well as the avoidance of negative results: for instance, armed with this knowledge one can prove Tychonoff's theorem in a few lines, whereas other proofs are significantly longer and more complicated (even when sequences suffice to describe the topology of the space!). In short, there are conceptual advantages to knowing "the truth" about convergence.

Intriguingly, there are two different theories of convergence which both successfully generalize the convergence of sequences in metric spaces: nets and filters. The theory of nets was developed by E.H. Moore and H.L. Smith [MS22]. In 1950 J.L. Kelley published a paper [Ke50a] which made some refinements on the theory, both substantial and cosmetric: (the term "net" appears for the first time in his paper). The prominent role of nets in his text [Ke] cemented the centrality of nets among American topologists. Then there is the rival theory of filters, discovered by Henri Cartan in 1937 amidst a Séminaire Bourbaki. Cartan successfully convinced his fellow Bourbakistes of the elegance and utility of the theory of filters, and Bourbaki's text [Bo] uses filters early and often. To this day most continental mathematicians retain a preference for the filter-theoretic language.

For the past fifty years or so, most topology texts have introduced at most one of nets and filters (possibly relegating the other to the exercises). As Gary Laison has pointed out, since both theories appear widely in the literature, this practice is a disservice to the student. The fact that the two theories are demonstrably equivalent – that is, one can pass from nets to filters and vice versa so as to preserve convergence, in a suitable sense – does not mean that one does not need to be conversant with both of them! In fact each theory has its own merits. The theory of nets is a rather straightforward generalization of the theory of sequences, so that if one has a sequential argument in mind, it is usually *a priori* clear how to phrase it in terms of nets. (In particular, one can make a lot of headway in functional analysis simply by doing a search/replace of "sequence" with "net.") Moreover, many complicated looking limiting processes in analysis can be expressed more simply and cleanly as convergence with respect to a net – e.g., the Riemann integral. One may say that the main nontriviality in the theory of nets is the notion of "subnet", which is more complicated than one at first expects: in particular, a subnet may have larger cardinality! The corresponding theory of filters is less straightforward initially, but most experts agree that it is eventually more penetrating. One advantage is that the filter-theoretic analogue of subnet is much more transparent: it is just set-theoretic containment. Filters have applications beyond just generalizing the notion of convergent sequences: in completions and compactifications, in Boolean algebra and in mathematical logic, where ultrafilters are arguably the single most important (and certainly the most elegant) single technical tool.

2. Sequences in Topological Spaces

In this section we develop the theory of convergence of sequences in arbitrary topological spaces, including an analysis of its limitations.

2.1. **Arbitrary topological spaces.** A sequence $\mathbf{x}$ in a topological space X **converges** to $x \in X$ if for every neighborhood U of x, $\mathbf{x}_n \in U$ for all sufficiently large n. Note that it would obviously be equivalent to say that all but finitely many terms of the sequence lie in any given neighborhood U of x, which shows that whether a sequence converges to x is independent of the ordering of its terms.[2]

Remark 2.1.1: The convergence of a sequence is a **topological notion**: i.e., if X, Y are topological spaces, $f : X \to Y$ is a homeomorphism, $\mathbf{x}_n$ is a sequence in X and x is a point of X, then $\mathbf{x}_n \to x$ iff $f(\mathbf{x}_n) \to f(x)$. In particular the theory of sequential convegence in metric spaces recalled in the preceding section applies verbatim to all metrizable spaces.

Tournant dangereuse: Let us not forget that in a metric space we have the notion of a **Cauchy sequence**, a sequence $\mathbf{x}_n$ with the property that for all $\epsilon > 0$, there exists $N = N(\epsilon)$ such that $m, n \geq N \implies d(\mathbf{x}_m, \mathbf{x}_n) < \epsilon$, together with the attendant notion of completness (i.e., that every Cauchy sequence be convergent) and completion. Being a Cauchy sequence is *not* a topological notion: let $X = (0,1)$, $Y = (1,\infty)$, $f : X \to Y$, $x \mapsto \frac{1}{x}$, and $\mathbf{x}_n = \frac{1}{n}$. Then $\mathbf{x}_n$ is a Cauchy sequence, but $f(\mathbf{x}_n) = n$ is not even bounded so cannot be a Cauchy sequence. (Indeed, the fact that boundedness is not a topological property is certainly relevant here.) This means that what is, for analytic applications, arguably the most important aspect of the theory – what is first semester analysis but an ode to the completeness of the real numbers? – cannot be captured in the topological context. However there is a remedy, namely Weil's notion of **uniform spaces**, which will be discussed later on.[3]

Example 2.1.2: Let X be a set with at least two elements endowed with the indiscrete topology. Let $\{\mathbf{x}_n\}$ be a sequence in X and $x \in X$. Then $\mathbf{x}_n$ converges to x.

Example 2.1.3: A sequence is **eventually constant** if there exists an $x \in X$ and an N such that $n \geq N \implies \mathbf{x}_n = x$; we say that x is the **eventual value** of the sequence (note that this eventual value is unique). In any topological space,

[2] This aspect of sequential convergence will *not* be preserved in the theory of nets.

[3] Here, by "later on" I meant "in some as yet unwritten notes". By coincidence, uniform space are *alluded to* later on in these notes, but a proper discussion is not to be found here: sorry!

an eventually constant sequence converges to its eventual value. However, such a sequence may have other limits as well, as in the above example.

Exercise 2.1.4: In a discrete topological space X, a sequence $\mathbf{x}_n$ converges to x iff $\mathbf{x}_n$ is eventually constant and x is its eventual value.

In particular the limit of a convergent sequence in a discrete space is unique. (Since discrete spaces are metrizable, by Remark 2.1.1 we knew this already.) The following gives a generalization:

Proposition 2.1. *In a Hausdorff space, a sequence converges to at most one point.*

Proof. If $\mathbf{x}_n \to x$ and $x' \neq x$, there exist disjoint neighborhoods N of x and N' of x'. Then only finitely many terms of the sequence can lie in N', so the sequence cannot converge to x. $\square$

Let $\iota : \mathbb{Z}^+ \to \mathbb{Z}^+$ be an increasing injection. If $\{x_n\}$ is a sequence in a space X, then so too is $\{x_{\iota(n)}\}$, a **subsequence** of $\{x_n\}$. Immediately from the definitions, if a sequence converges to a point x then every subsequence converges to x. On the other hand, a divergent sequence may admit a convergent subsequence. We say that x is a **limit point** of a sequence $\mathbf{x}_n$ if every neighborhood N of x contains infinitely many terms from the sequence.

A space X is **first countable at x** $\in X$ if there is a countable base at x. A space is **first countable** if it is first countable at each of its points.

Proposition 2.2. *Metric (hence also metrizable) spaces are first countable.*

Proof. Let (X, d) be a metric space, and let x be any point of X. Then the set $\{B(x, \frac{1}{n})\}_{n\in\mathbb{Z}^+}$ of $\frac{1}{n}$-balls at x is a countable neighborhood base at x. $\square$

The countable base of $\frac{1}{n}$-balls at x in (X, d) is **nested**: $N_1 \supset N_2 \supset \ldots$. This is not particular to metric spaces: in any topological space X, if $\{N_n\}$ is a countable base at $x \in X$, then $N'_n = \cap_{i=1}^n N_i$ is a nested countable base at x. This observation underlies the role that sequences play in the topology of a first countable space.

Proposition 2.3. *Let X be a first countable space and $Y \subset X$. Then $\overline{Y}$ is the set of all limits of sequences from Y.*

Proof. Suppose y_n is a sequence of elements of Y converging to x. Then every neighborhood N of x contains some $y_n \in Y$, so that $x \in \overline{Y}$. Conversely, suppose $x \in \overline{Y}$. If X is first countable at x, we may choose a nested collection $N_1 \supset N_2 \supset \ldots$ of open neighborhoods of x such that every neighborhood of x contains some N_n. Each N_n meets Y, so choose $y_n \in N_n \cap Y$, and y_n converges to y. $\square$

Proposition 2.4. *Let f be a map of sets between the topological spaces X and Y. Assume that X is first countable. TFAE:*
a) f is continuous.
b) If $\mathbf{x}_n \to x$, $f(\mathbf{x}_n) \to f(x)$.

Proof. a) $\implies$ b): Let V be any open neighborhood of $f(x)$; by continuity there exists an open neighborhood U of x such that $f(U) \subset V$. Since $\mathbf{x}_n \to x$, there exists N such that $n \geq N$ implies $\mathbf{x}_n \in U$, so that $f(\mathbf{x}_n) \in V$. Therefore $f(\mathbf{x}_n) \to f(x)$.

b) $\implies$ a): Suppose f is not continuous, so that there exists an open subset V

of Y with $U = f^{-1}(V)$ not open in X. More precisely, let x be a non-interior point of U, and let $\{N_n\}$ be a nested base of open neighborhoods of x. By non-interiority, for all n, choose $\mathbf{x}_n \in N_n \setminus U$; then $\mathbf{x}_n \to x$. By hypothesis, $f(\mathbf{x}_n) \to f(x)$. But V is open, $f(x) \in V$, and $f(\mathbf{x}_n) \in Y \setminus V$ for all n, a contradiction. $\square$

Proposition 2.5. *A first countable space in which each sequence converges to at most one point is Hausdorff.*

Proof. Suppose not, so there exist distinct points x and y such that every neighborhood of x meets every neighborhood of Y. Let U_n be a nested neighborhood basis for x and V_n be a nested neighborhood basis for y. By hypothesis, for all n there exists $\mathbf{x}_n \in U_n \cap V_n$. Then $\mathbf{x}_n \to x$, $\mathbf{x}_n \to y$, contradiction. $\square$

Proposition 2.6. *Let $\{x_n\}$ be a sequence in a first countable space. TFAE:*
a) x is a limit point of the sequence.
b) There exists a subsequence converging to x.

Proof. a) $\implies$ b): Take a nested neighborhood basis N_n of x, and for each $k \in \mathbb{Z}^+$ choose successively a term $n_k > n_{k-1}$ such that $x_{n_k} \in N_k$. Then $x_{n_k} \to x$. The converse is almost immediate and does not require first countability. $\square$

The following example shows that the hypothesis of first countability is necessary for each of the previous three results.

Example 2.1.5: Let X be an uncountable set. The family of subsets $U \subset X$ with countable complement together with the empty set forms a topology on X, the **cocountable topology**. This is a non-discrete topology (since X is uncountable). In fact it is not even Hausdorff, if N_x and N_y are any two neighborhoods of points x and y, then $X \setminus N_x$ and $X \setminus N_y$ are countable, so $X \setminus (N_x \cap N_y) = (X \setminus N_x) \cup (X \setminus N_y)$ is uncountable and $N_x \cap N_y$ is nonempty. However, in this topology $\mathbf{x}_n \to x$ iff $\mathbf{x}_n$ is eventually constant with eventual value x. Indeed, let $\mathbf{x}_n$ be a sequence for which the set of n such that $\mathbf{x}_n \neq x$ is infinite. Then $X \setminus \{x_n \neq x\}$ is a neighborhood of x which omits infinitely many terms $\mathbf{x}_n$ of the sequence, so $\mathbf{x}_n$ does not converge to x. This also implies that the set of all limits of sequences from a subset Y is just Y itself, whereas for any uncountable Y, $\overline{Y} = X$.

Exercise 2.1.6: A point x of a topological space is **isolated** if $\{x\}$ is open.
a) If x is isolated, and $\mathbf{x}_n \to x$, then $\mathbf{x}_n$ is eventually constant with limit x.
b) Note that Example 2.1.3 shows that the converse is false in general. Show however, that if X is first countable and x is not isolated, then there exists a non-eventually constant sequence converging to x.

2.2. **Sequential spaces.** Note that the hypothesis of first countability appeared as a sufficient condition in most of our results on the topological adequacy of convergent sequences. It is natural to ask to what extent it is necessary.

To explore this let us define the **sequential closure** $\mathrm{sc}(Y)$ of a subset Y of X to be the set of all limits of convergent sequences from Y. We have just seen that $sc(Y) \subset \overline{Y}$ in any space, $\mathrm{sc}(Y) = \overline{Y}$ in a first countable space, and in general we may have $\mathrm{sc}(Y) \neq \overline{Y}$.

Generalizing first countability, one calls a space **Fréchet** if $\operatorname{sc}(Y) = \overline{Y}$ for all Y. However, a yet weaker condition is in some ways more interesting. Namely, define a space to be **sequential** if sequentially closed subsets are closed. Here are some easy facts:

(i) Closed subspaces of sequential spaces are sequential.
(ii) A space is Fréchet iff every subspace is sequential.
(iii) A space is sequential iff $\operatorname{sc}(Y) \setminus Y \neq \emptyset$ for every nonclosed subset Y.
(iv) Let $f : X \to Y$ be a map between topological spaces. If X is sequential, then f is continuous iff $\mathbf{x}_n \to x \implies f(\mathbf{x}_n) \to f(x)$.

Next we note that in any space, $A \mapsto \operatorname{sc}(A)$ satisfies the three Kuratowski closure axioms (KC1), (KC2), (KC4), but not in general (KC3). As the proof of [**Topological Spaces**, Thm. 1] shows, the sequentially closed sets therefore satisfy the axioms (CTS1)-(CTS3) for the closed sets of a new, finer topology τ' on X.

Consider next the prospect of iterating the sequential closure. If X is not sequential, there exists some nonclosed subset A whose sequential closure is equal to A itself, and then no amount of iteration will bring the sequential closure to the closure. Conversely, if X is sequential but not Fréchet, then for some nonclosed subset A of X we have A is properly contained in $\operatorname{sc}(A)$ which is properly contained in $\operatorname{sc}(\operatorname{sc}(A))$. For any ordinal number α, we can define the α-iterated sequential closure sc_α, by $\operatorname{sc}_{\alpha+1}(A) = \operatorname{sc}(\operatorname{sc}_\alpha(A))$, and for a limit ordinal β we define

$$\operatorname{sc}_\beta(A) = \bigcup_{\alpha<\beta} \operatorname{sc}_\alpha(A).$$

There is then some ordinal α such that $\operatorname{sc}_\alpha(A) = \overline{A}$ for all subsets A of X. The least such ordinal is called the **sequential order**, and is an example of an **ordinal invariant** of a topological space.

These ideas have been studied in considerable detail, notably by R.M. Dudley and S.P. Franklin [Du64], [Fr65], [AF68].

One would think that there could arise, in practice, situations in which one was naturally led to consider sequential closure and not closure. (In fact, it seems to me that this is the case in the theory of **equidistribution** of sequences. But not being too sure of myself, I will say nothing further about it here.) However, we shall not pursue the matter further here, but rather turn next to two ways of "repairing" the notion of convergence by working with more general objects than sequences.

3. Nets

3.1. **Nets and subnets.**

On a set I equipped with a binary relation $\leq$, consider the following axioms:

(PO1) For all $i \in I$, $i \leq i$. (reflexivity).
(PO2) For all $i,\ j, k \in I$, $i \leq j,\ j \leq k$ implies $i \leq k$. (transitivity).
(PO3) If $i \leq j$ and $j \leq i$, then $i = j$ (anti-symmetry).

(D) For $i,\ j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

If $\leq$ satisfies (PO1), (PO2) and (PO3), it is called a **partial ordering**. We trust that this is a familiar concept. If $\leq$ (PO1) and (PO2) it is called a **quasi-ordering**.[4] Finally, a relation which satisfies (PO1), (PO2) and (D) is said to be **directed**, and a *nonempty set* I endowed with $\leq$ is called a **directed set**.

Example 3.1.1: A nonempty set I endowed with the "maximal" (discrete??) relation $I \times I$ – i.e., $x \leq y$ for all $x, y \in I$ is directed, but not partially ordered if I has more than one element.

Example 3.1.2: Any totally ordered set is a directed set; in particular the positive integers with their standard ordering form a directed set.

A subset J of a directed set I is **cofinal** if for all $i \in I$, there exists $j \in J$ such that $j \geq i$. For instance, a subset of $\mathbb{Z}^+$ is cofinal iff it is infinite. A cofinal subset of a directed set is itself directed.

Example 3.1.3: The neighborhoods of a point x in a topological space form a directed (and partially ordered) set under reverse inclusion. More explicitly, we define $N_1 \leq N_2$ iff $N_1 \supset N_2$. A cofinal subset is precisely a neighborhood basis.

If X has a countable basis at x, then we saw that we could take a nested neighborhood basis. In other words, the directed set of neighborhoods has a cofinal subset which is order isomorphic to the positive integers $\mathbb{Z}^+$, and this structure was the key to the efficacy of sequential convergence in first countable spaces. This suggests modiyfying the definition of convergence by replacing sequences by functions with domain in an aribtrary directed set:

A **net** $\mathbf{x} : I \to X$ in a set X is a mapping from a directed set I to X.

Some further net-theoretic (but not yet topological) terminology: a net $\mathbf{x} : I \to X$ is **eventually in** a subset A of X if there exists $i \in I$ such that for all $j \geq i$, $x_j \in A$. Moreover, $\mathbf{x}$ is **cofinally** in A if the set of all i such that $x_i \in A$ is cofinal in I.

Exercise 3.1.4: For a net $\mathbf{x} : I \to X$ and a subset A of X, TFAE:
(i) $\mathbf{x}$ is cofinal in A.
(ii) $\mathbf{x}$ is *not* eventually in $X \setminus A$.
Now suppose that we have a net $x_\bullet : I \to X$ in a topological space X. We say that $x_\bullet$ **converges** to $x \in X$ – and write $\mathbf{x} \to x$ or $x_i \to x$ – if for every neighborhood U of x, there is an element $i \in I$ such that for all $j \geq i$, $x_j \in U$. In other words, $x_i \to x$ iff $\mathbf{x}$ is eventually in every neighborhood of x. Moreover, we say that x is a **limit point** of $\mathbf{x}$ if $\mathbf{x}$ is cofinally in every neighborhood of x.

Exercise 3.1.5: Stop and check that for nets with $I = \mathbb{Z}^+$ this reduces to the definition of limit and limit point for sequences given in the previous section.

[4] Alternate terminology: **preordering**.

Now the following result almost proves itself:

Proposition 3.1. *In a topological space X, the closure of any subset S is the set of limits of convergent nets of elements of S.*

Proof. First, if x is the limit of a net $\mathbf{x}$ of elements of S, then if x were not in $\overline{S}$ there would exist an open neighborhood U of x disjoint from S, but the definition of a net ensures that the set of $i \in I$ for which $x_i \in U \cap S$ is nonempty, a contradiction. On the other hand, assume that $x \in \overline{S}$, and let I be the set of open neighborhoods of x. For each i, select any $x_i \in i \cap S$. That the net x_i converges to x is a tautology: each open neighborhood U of x correponds to some $i \in I$, and for all $j \geq i$ – i.e., for all open neighborhoods $V = V(j) \subset U = U(i)$ – we do indeed have $x_j \in V$. $\square$

Proposition 3.2. *For a map f between the topological spaces X and Y, TFAE:*
a) f is continuous.
b) If $\mathbf{x}$ is a net converging to x, then $f(\mathbf{x})$ is a net converging to $f(x)$ in Y.

Proposition 3.3. *A space is Hausdorff iff each net converges to at most one point.*

Exercise 3.1.6: Prove Propositions 3.2 and 3.3.

We would now like to give the "net-theoretic analogue" of Proposition 2.6. Its statement should clearly be the following:

Tentative Proposition. *Let $\mathbf{x}$ be a net in a topological spcae. TFAE:*
a) x is a limit point of $\mathbf{x}$.
b) There exists a subnet converging to x.

Of course, in order to make proper sense of this we need to define "subnet": how to do this? It is tempting to define a subnet of $\mathbf{x} : I \to X$ as the net obtained by restricting x to a cofinal subset of I. (At any rate, this is what I would have guessed.) However, with this definition, a subnet of a sequence is nothing else than a subsequence, and although this may sound appealing initially, it would mean that Proposition 2.6 is true without the assumption of first countability. This is not the case, as the following example shows.

Example 3.1.7 (Arens): Let $X = \mathbb{Z}^+ \times \mathbb{Z}^+$, topologized as follows: every one-point subset except $(0,0)$ is open, and the neighborhoods of $(0,0)$ are those subsets N containing $(0,0)$ for which there exists an M such that $m \geq M \implies \{n \mid (m,n) \not\in N\}$ is finite: that is, N contains all but finitely many of the elements of all but finitely many of the columns $M \times \mathbb{Z}^+$ of X. Then X is a Hausdorff space in which no sequence in $X \setminus \{(0,0)\}$ converges to $(0,0)$. Moreover, there is a sequence $\mathbf{x}_n \in X \setminus \{(0,0)\}$ which has $(0,0)$ as a limit point, but by the above there is no subsequence which converges to $(0,0)$.

So we define a **subnet** of a net $\mathbf{x} : I \to X$ to be a net $\mathbf{y} : J \to X$ for which there exists an order homomorphism $\iota : J \to I$ (i.e., $j_1 \leq j_2 \implies \iota(j_1) \leq \iota(j_2)$) with $\mathbf{y} = \mathbf{x} \circ \iota$ such that $\iota(J)$ is cofinal in I. This differs from the expected definition in that ι is not required to be an injection. Indeed, J may have larger cardinality than I, and this is an important feature of the definition.

Exercise 3.1.8: Let J and I be a directed sets. A function $\iota : J \to I$ is said to be **cofinal** if for all $i \in I$ there exists $j \in J$ such that $j' \geq j \implies \iota(j') \geq i$. Show that the order homomorphism ι required in the definition of subnet is a cofinal function.

Remark 3.1.9: Indeed, many treatments of the theory (e.g. Kelley's) require only that the function ι be cofinal, which gives rise to a more inclusive definition of a subnet. The two definitions lead to exactly the same results, so the issue of which one to adopt is purely a matter of taste. Our perspective here is that by restricting as we have to "order-preserving subnets", results of the form "There exists a subnet such that..." become (in the formal sense) slightly stronger.[5]

Exercise 3.1.10: Let $\mathbf{y}$ be a subnet of $\mathbf{x}$ and $\mathbf{z}$ be a subnet of $\mathbf{y}$. Show that $\mathbf{z}$ is a subnet of $\mathbf{x}$.

In order to make progress, we need the following key technical result.

Lemma 3.4. *(Kelley's Lemma) Let $x : I \to X$ be a net in the topological space X, and $\mathcal{A}$ a family of subsets of X. We assume:*
(i) For all $A \in \mathcal{A}$, $I_A := \{i \in I \mid x_i \in A\}$ is cofinal in A.
(ii) The intersection of any two elements of $\mathcal{A}$ contains an element of A.
Then there is a subnet y of x which is eventually in A for all $A \in \mathcal{A}$.

Proof. Property (ii) implies that the family $\mathcal{A}$ is directed by $\supset$. Let J be the set of all pairs (i, A) such that $i \in I$, $A \in \mathcal{A}$ and $x_i \in A$, endowed with the induced ordering from the product $I \times \mathcal{A}$. It is easy to see that J is a directed set. Indeed: For $(i, A), (i', A')$ in J, we may choose first $A'' \subset A' \cap A''$ and then $i'' \in I$ such that $i'' \geq i$, $i'' \geq i'$ and $x_{i''} \in A''$, and then (i'', A'') is an element of J dominating (i, A) and (i', A'). Moreover, the natural map $\iota : J \to I$ given by $(i, A) \mapsto i$ is an order homomorphism. Since $I_A \times \{A\} \subset J$ and I_A is cofinal for all $A \in \mathcal{A}$, $\iota(J)$ is cofinal in I, so that $y := x \circ \iota$ is a subnet of x. Fix $A \in \mathcal{A}$ and choose $i \in I$ such that $x_i \in A$. If $(i', A') \geq (i, A)$, then $x_{i'} \in A' \subset A$, so that $y_{(i', A')} = x_{i'} \in A$, and y is eventually in A. $\square$

Now let us restate and prove our tentative proposition about subnets.

Proposition 3.5. *Let $\mathbf{x}$ be a net in a topological spcae. TFAE:*
(i) x is a limit point of $\mathbf{x}$.
(ii) There exists a subnet converging to x.

Proof. (i) $\implies$ (ii): Let $x \in X$ be a limit point of a net $x_\bullet$. Applying Lemma 3.4 to the family of all neighborhoods of x, we get a subnet of $x_\bullet$ converging to x.
$\neg$ (i) $\implies$ $\neg$ (ii): If x is not a limit point of $x_\bullet$, then there is a neighborhood N of x such that I_N is not cofinal in I, meaning that I is eventually in $X \setminus N$. It follows that every subnet is eventually in $X \setminus N$ and thus no subnet converges to x. $\square$

Exercise 3.1.11:
a) Define an "eventually constant net."
b) Show that for a topological space X and $x \in X$, TFAE:

[5] After gaining inspiration from the theory of filters, we will offer a definition of subnet which is more inclusive than Kelley's and seems simpler: it does not require an auxiliary function ι.

(i) x is an isolated point of X.
(ii) Every net converging to x is eventually constant.
c) Show: a nondiscrete space carries a convergent but not eventually constant net.

Exercise 3.1.12: Let $\mathbf{x}$ be a net on a set X, $\mathbf{y}$ a subnet of X, x a point of X and A a subset of X.
a) If $\mathbf{x}$ is eventually in A, then $\mathbf{y}$ is eventually in A.
b) If $\mathbf{x} \to x$, then $\mathbf{y} \to x$.
c) If $\mathbf{y}$ is cofinally in A, so is $\mathbf{x}$.
d) If x is a limit point of $\mathbf{y}$, it is also a limit point of $\mathbf{x}$.

3.2. **Two examples of nets in analysis.**

Example 3.2.1: Let $A = \{a_i\}$ be an indexed family of real numbers, i.e., a function from a naked set S to $\mathbb{R}$. Can we make sense of the infinite series $\sum_{i \in S} a_i$? Note that we are assuming no ordering on the terms of the series, which may look worrisome, since in case $S = \mathbb{Z}^+$ it is well-known that the convergence of the series (and its sum) *will* in general depend upon the ordering relation on I we use to form the sequence of partial sums.

Nevertheless, there is a nice answer. We say that the series $\sum_{i \in S} a_i$ **converges unconditionally** to $a \in \mathbb{R}$ if: for all $\epsilon > 0$, there exists a finite subset $J(\epsilon)$ of S such that for all finite subsets $J(\epsilon) \subset J \subset S$, we have $|a - \sum_{i \in J} a_i| < \epsilon$.

Exercise 3.2.2:
a) Show that if $\sum_{i \in I} a_i$ is unconditionally convergent, then the set of indices $i \in I$ for which $a_i \neq 0$ is at most countable.
b) Suppose $I = \mathbb{Z}^+$. Show that a series converges unconditionally iff it converges absolutely, i.e., iff $\sum_{i=1}^{\infty} |a_i| < \infty$.
c) Define unconditional and absolute convergence of series in any real Banach space. Show that absolute convergence implies unconditional convergence, and find an example of a Banach space in which there exists an unconditionally convergent series which is not absolutely convergent.[6]

The point is that this "new" type of limiting operation can be construed as an instance of net convergence. Namely, let $I(S)$ be the set of all finite subsets J of S, directed under containment. Then given $\mathbf{a} : S \to \mathbb{R}$, we can define a net $\mathbf{x}$ on $I(S)$ by $J \mapsto \sum_{i \in J} a_i$. Then the unconditional convergence of the series is equivalent to the convergence of the net $\mathbf{x}$ in $\mathbb{R}$.

Exercise 3.2.3: Suppose that we had instead decided to define $\sum_{i \in S} a_i$ converges unconditionally to a as: for all $\epsilon > 0$, there exists $N = N(\epsilon)$ such that for all finite subsets J of S with $\# J \geq N$ we have $|a - \sum_{i \in J} a_i| < \epsilon$.
a) Show that this is again an instance of net convergence.
b) Is this equivalent to the definition we gave?

Example 3.2.4: The collection of all tagged partitions $(\mathcal{P}, x_i^*)$ of $[a, b]$ forms a

[6]In fact, the celebrated Dvoretzky-Rogers theorem asserts that a Banach spaces admits an unconditionally but nonabsolutely convergent series iff it is infinte- dimensional.

directed set, under the relation of inclusion $\mathcal{P} \subset \mathcal{P}'$ ("refinement"). A function $f : [a,b] \to \mathbb{R}$ defines a net in $\mathbb{R}$, namely

$$(\mathcal{P}, x_i^*) \mapsto R(f, \mathcal{P}, x_i^*),$$

the latter being the associated Riemann sum.[7] The function f is Riemann-integrable to L iff the net converges to L.

Examples like these motivated Moore and Smith to develop their generalized convergence theory.

3.3. **Universal nets.** A net $\mathbf{x} : I \to X$ in a set X is said to be **universal**[8] if for any subset A of X, x is either eventually in A or eventually in $X \setminus A$.

Exercise 3.3.1: Show that a net in X is universal iff whenever it is cofinally in a subset A of X, it is eventually in A.

Exercise 3.3.2: Let $\mathbf{x} : I \to X$ be a net, and let $f : X \to Y$ be a function.
a) Show that if $\mathbf{x}$ is universal, so is the induced net $f(\mathbf{x}) = f \circ \mathbf{x}$.
b) Show that the converse need not hold.

Exercise 3.3.3: Show that any subnet of a universal net is universal.

Example 3.3.4: An eventually constant net is universal.

Less trivial examples are difficult to come by. Note that a convergent net need not be universal: for instance, take the convergent sequence $\mathbf{x}_n = \frac{1}{n}$ in $[0,1]$ and $A = \{1, \ \frac{1}{3}, \ \frac{1}{5}, \ldots\}$. Then the sequence is cofinal in both A and its complement so is not eventually in either one. Indeed, the same argument shows that a *sequence* which is universal is eventually constant.

Nevertheless, one has the following result.

Theorem 3.6. *(Kelley) Every net admits a universal subnet.*

Proof. Let $\mathbf{x}$ be a net in X, and consider all collections $\mathcal{A}$ of subsets of X such that:

(i) $Y_1, \ Y_2 \in \mathcal{A} \implies Y_1 \cap Y_2 \in \mathcal{A}$.
(ii) $Y_1 \in \mathcal{A}, \ Y_2 \supset Y_1 \implies Y_2 \in \mathcal{A}$.
(iii) $Y \in \mathcal{A} \implies \mathbf{x}$ is cofinal in Y.

The set of all such families is nonempty, since $\mathcal{A} = \{X\}$ is one. The collection of such families is therefore a nonempty poset under the relation $\mathcal{A}_1 \leq \mathcal{A}_2$ if $\mathcal{A}_1 \subset \mathcal{A}_2$. The union of a chain of such families is is immediately checked to be such family, so Zorn's Lemma entitles us to a family $\mathcal{A}$ which is not properly contained in any other such family. We claim that such an $\mathcal{A}$ has the following additional property: for any $A \subset X$, either $A \in \mathcal{A}$ or $X \setminus A \in \mathcal{A}$.

Indeed, suppose first that for every $Y \in \mathcal{A}$, $\mathbf{x}$ is cofinal in $A \cap Y$. Then the family

[7] Moreover, all of the standard variations on the definitio of Riemann integrability – e.g. upper and lower sums – can be similarly described in terms of convergence of nets.

[8] Alternate terminology: **ultranet**.

$\mathcal{A}'$ of sets containing $A \cap Y$ for some $Y \in \mathcal{A}$ satisfies (i), (ii) and (iii) and contains $\mathcal{A}$, so by maximality $\mathcal{A}' = \mathcal{A}$ and hence $A = A \cap X$ is in $\mathcal{A}$ and $\mathbf{x}$ is cofinal in A.

So we may assume there exists $Y \in \mathcal{A}$ such that $\mathbf{x}$ is not cofinal in $A \cap Y$, i.e., $\mathbf{x}$ is eventually in (so *a fortiori* is cofinal in) $X \setminus (A \cap Y)$. Then by the previous case, $X \setminus (A \cap Y) \in \mathcal{A}$; by (ii) so too is

$$Y \cap (X \setminus A \cap Y) = Y \setminus (A \cap Y),$$

and then by (ii) we get $X \setminus A \in \mathcal{A}$.

Now we apply Kelley's Lemma (Lemma 3.4) to the net $\mathbf{x} : I \to X$ and the family $\mathcal{A}$: we get a subnet $y_\bullet$ which is eventually in each $A \in \mathcal{A}$. Since $\mathcal{A}$ has the property that for all A, either A or $X \setminus A$ lies in $\mathcal{A}$, this subnet is universal. $\square$

At this point, the reader who is not wondering "What on earth is the point of universal nets?" is either a mathematical genius, has seen the material before or is pathologically uncurious. The following results provide a hint:

Proposition 3.7. *For a universal net* $\mathbf{x}$ *in a topological space, and* $x \in X$*, TFAE:*
(i) x *is a limit point of* $\mathbf{x}$.
(ii) $\mathbf{x} \to x$.

Proof. Of course (ii) $\implies$ (i) for all nets. Conversely, if x is a limit point of $\mathbf{x}$, then $\mathbf{x}$ is eventually in every neighborhood U of x. But then, by Exercise 3.3.1, universality implies that $\mathbf{x}$ is eventually in N. So $\mathbf{x} \to x$. $\square$

Proposition 3.8. *Let* X *be a topological space. The following are equivalent:*
(i) Every net in X *admits a convergent subnet.*
(ii) Every net in X *has a limit point.*
(iii) Every universal net in X *is convergent.*

Proof. By Proposition 3.5 (i) $\implies$ (ii); by Proposition 3.7 (ii) $\implies$ (iii); and by Theorem 3.6 (iii) $\implies$ (i). $\square$

Recall that in the special case of metric spaces these conditions hold with "net" replaced by "sequence", and moreover they are equivalent to the **Heine-Borel condition** that every open cover admits a finite subcover (Theorem 1.7, which we have not yet proved). We shall now see that, for any topological space, our net-theoretic analogues of Proposition 3.8 are equivalent to the Heine-Borel condition.

4. Convergence and (Quasi-)Compactness

4.1. Net-theoretic criteria for quasi-compactness.

Definition: A family $\{U_i\}_{i \in I}$ of subsets of a set X is said to **cover** X if $X = \bigcup_{i \in I} U_i$. A family $\{F_i\}_{i \in I}$ of subsets of a set X is said to satisfy the **finite intersection property** (FIP) if for every finite subset $J \subset I$, $\cap_{i \in J} F_i \neq \emptyset$.

Theorem 4.1. *For a topological space* X*, TFAE:*
a) Every net in X *admits a convergent subnet.*
b) Every net in X *has a limit point.*
c) Every universal net in X *is convergent.*
d) (Heine-Borel condition) For every cover of X *by open subsets* $\{U_i\}_{i \in I}$*, there exists a finite subset* $J \subset I$ *such that* $\bigcup_{i \in J} U_i = X$*. ("Every open cover admits a*

finite subcover.")
e) For every family $\{F_i\}_{i\in I}$ *of closed subsets satisfying the finite intersection property,* $\cap_{i\in I} F_i \neq \emptyset$.
A space satisfying these equivalent conditions is said to be ***quasi-compact****.*

Proof. The equivalence of a), b) and c) has already been shown. The equivalence of d) and e) is "due to de Morgan": property d) becomes property e) upon setting $F_i = X \setminus U_i$, and conversely. Thus it suffices to show b) $\implies$ e) $\implies$ b).

Assume b), and let $\{F_i\}_{i\in I}$ be a family of closed subsets satisfying the finite intersection property. Then the index set I is directed under reverse inclusion. For each $i \in I$, choose any $x_i \in F_i$; the assignment $i \mapsto x_i$ is then a net $\mathbf{x}$ in X. Let x be a limit point of $\mathbf{x}$, and assume for a contradiction that there exists i such that x does not lie in F_i. Then $x \in U_i = X \setminus F_i$, and by definition of limit point there exists some index $j > i$ such that $x_j \in U_i$. But $j > i$ means $F_j \subset F_i$, so that $x_j \in F_j \cap U_i \subset F_i \cap U_i = (X \setminus U_i) \cap U_i = \emptyset$, contradiction! Therefore $x \in \cap_{i\in I} F_i$.

Now assume e) and let $\mathbf{x} : I \to X$ be a net in X. For each $i \in I$, define $F_i = \overline{\{x_j \mid j \geq i\}}$. Since directedness implies that given any finite subset J of I there exists some $i \in I$ such that $i \geq j$ for all $j \in J$, the family $\{F_i\}_{i\in I}$ of closed subsets satisfies the finite intersection condition. Thus by our assumption there exists $x \in \cap_{i\in I} F_i$. Let U be any neighborhood of x and take any $i \in I$. Then $x \in \overline{F_i}$, so that $F_i \cap U$ is nonempty. In other words, there exists $j \geq i$ such that $x_j \in U$, and this means that $\mathbf{x}$ is cofinal in U. Since U was arbitrary, we conclude that x is a limit point of $\mathbf{x}$. □

Exercise 4.1.1: If X is quasi-compact and $f : X \to Y$ is continuous, then $f(X)$ is quasi-compact.

Remark 4.1.2: Following N. Bourbaki, we reserve the term **compact** for a *Hausdorff* space satisfying the conditions of Theorem 4.1.

4.2. **Compactness in metrizable spaces.**

Now would seem to be an appropriate time to discuss compactness in metrizable spaces. In order to do so we shall rather briskly introduce some concepts and terminology that will be discussed in more detail later on, but with which we imagine most readers have some prior acquaintance.

First, let Y be a subset of X. We say that Y is covered by a family $\{U_i\}$ of subsets of X if $Y \subset \bigcup_i U_i$. This gives us the notion of a **quasi-compact subset** Y of X, i.e., a subset for which every open cover admits a finite subcover. There is also the notion of a subset being quasi-compact in the induced (subspace) topology[9]. Fortunately these two notions coincide:

Proposition 4.2. *Let* Y *be a subset of a topological space* X*. TFAE:*
a) Y *is a quasi-compact subset.*
b) Y *is quasi-compact in the subspace topology.*

Proof. a) $\implies$ b): Suppose Y is a quasi-compact subset and V_i is a cover by open subsets of Y. Then $V_i = U_i \cap Y$ for U_i open in X and $\{U_i\}$ is an open cover of Y

[9]Recall that the subspace topology on $Y \subset X$ is the one in which the open sets are $U \cap Y$ for U open in X.

as a subset of X, so by assumption it admits a finite subcover: there exists a finite subset $J \subset I$ such that $\bigcup_{i \in J} U_i \supset Y$. Then $\bigcup_{i \in J} V_i = Y$. The converse is similar and left to the reader. $\square$

Proposition 4.3. *A closed subset Y of a quasi-compact space X is quasi-compact.*

Proof. By the previous result it is enough to show that Y is quasi-compact as a subset of X, so let $\{U_i\}$ be a cover of Y by open subsets of X. Then $\{U_i\} \cup X \setminus Y$ is an open cover of X, which, by quasi-compactness of X admits a finite subcover. Removing $X \setminus Y$ from this finite cover, if necessary, we get a finite subcover for Y. $\square$

Proposition 4.4. *A topological space which is quasi-compact and discrete is finite.*

Proof. If X is discrete, $\{x\}_{x \in X}$ is an open cover of X without a proper subcover, so X can only be quasi-compact if the cover is already finite. $\square$

There are many properties which are equivalent to quasi-compactness on the class of metrizable spaces, but not for arbitrary spaces: indeed, some of them are not even implied by quasi-compactness. The list follows:

A space is **sequentially compact** if each sequence has a convergent subsequence.

A space X is **limit point compact** if each infinite subset has a limit point in X.

A space is **countably compact** if each countable open cover has a finite subcover.

Proposition 4.5.
a) A quasi-compact space is countably compact and limit point compact.
b) A sequentially compact space is countably compact.
c) A countably compact space is limit point compact.

Proof. a) Evidently quasi-compactness implies countable compactness. Now suppose A is an infinite subset of X with no accumulation point in X. Then A is closed, since a point lying in $\overline{A} \setminus A$ would be an accumulation point of A, so by Proposition 4.3 A is itself quasi-compact. Moreover, for $a \in A$, since a is not an accumulation point of A there exists a neighborhood U of x such that $U \cap A = \{x\}$. Therefore in the subspace topology A is both quasi-compact and discrete, hence by Proposition 4.4 A is finite, contradiction.

b) According to de Morgan, countable compactness is equivalent to the assertion that every countable family of closed subsets satisfying the finite intersection property has nonempty intersection. Replacing F_i by $F_1 \cap \ldots \cap F_n$ if necessary, this is turn equivalent to the fact that any nested sequence $F_1 \supset F_2 \supset \ldots$ of nonempty closed subsets has nonempty intersection. Let $\{F_i\}$ be such a sequence, and choose $x_i \in F_i$. By sequential compactness, there exists a subsequence x_{n_i} converging to x. Suppose that for some i we had $x \in X \setminus F_i$. Let j be sufficiently large such that $n_j \geq i$ and $x_{n_j} \in X \setminus F_i$. Then $x_{n_j} \in F_{n_j} \subset F_i$, contradiction.
c) Suppose A is an infinite subset of X without a limit point in X; by passing to a subset if necessary we may assume $A = \{a_i\}$ is countable. By part a) A is closed and discrete, so putting $A_i = \{a_i, a_{i+1}, \ldots\}$ we get a nested sequence of closed subsets of X satisfing the finite intersection property but with $\bigcap_{i \in \mathbb{Z}^+} A_i = \emptyset$. $\square$

Remark 4.2.1: None of the other implications between the four properties hold on the class of all topological spaces.

Proposition 4.6. *Consider the following properties of a topological space X:*
(i) X has a countable base.
(ii) X has a countable dense subset. ("X is separable.")
(iii) Every open cover of X has a countable subcover. ("X is Lindelöf.")
Then: a) (i) $\implies$ (ii), and (i) $\implies$ (iii).
b) If X is metrizable then (ii) $\implies$ (i), and (iii) $\implies$ (i), and thus all three properties are equivalent.

Proof. a) (i) $\implies$ (ii): Choosing one point x_i in each element U_i of a countable base gives a countable dense subset.
(i) $\implies$ (iii): Moreover, if $\{U_n\}_{n\in\mathbb{Z}^+}$ is a countable base and $\{V_\alpha\}_{\alpha\in I}$ is an open cover of X, then for each $x \in X$, there exists $i \in \mathbb{Z}^+$ and α such that $x \in U_i \subset V_\alpha$. This defines a (countable!) subset S of $\mathbb{Z}^+$, namely the set of all i such that U_i is contained in some V_α, and choosing for each $i \in S$ some α_i such that $U_i \subset V_\alpha$, $\bigcup_{i\in S} V_{\alpha_i} = X$.
b) Suppose X is metrizble.
(ii) $\implies$ (i): If $\{x_n\}_{n\in\mathbb{Z}^+}$ is a countable dense subset, then $\{B(x_n, \frac{1}{m})\}_{m,n\in\mathbb{Z}^+}$ is a countable base.
(iii) $\implies$ (i): Suppose every cover admits a countable subcover. In particular, for each $n \in \mathbb{Z}^+$ the collection of all open balls of radius $\frac{1}{n}$ has a countable subcover $B(x_{n,i}, \frac{1}{n})$. The countable subset $\{x_{n,i}\}_{(n,i)\in\mathbb{Z}^+\times\mathbb{Z}^+}$ is then a base for X. Indeed, let $y \in X$ and $\epsilon > 0$, and choose n with $\frac{2}{n} < \epsilon$. Then there exists an i such that $y \in B(x_{n,i}, \frac{1}{n})$ and hence $B(x_{n,i}, \frac{1}{n}) \subset B(y, \epsilon)$. □

Theorem 4.7. *a) In a first countable space, limit point compactness implies sequential compactness.*
b) In a metrizable space, sequential compactness implies quasi-compactness, and hence quasi-compactness, sequential compactness, limit point compactness, and countable compactness are all equivalent properties.

Proof. Suppose first that X is first countable and limit point compact, and let $\mathbf{x}$ be a sequence in X. If the image of the sequence is finite, we can extract a constant, hence convergent, subsequence. Otherwise the image is an infinite subset of X, which (since quasi-compactness implies limit point compactness) has a limit point x, which is in particular a limit point of the sequence. Then, as in any first countable space, there is a subsequence converging to x.

Now suppose X is sequentially compact. For each $n \in \mathbb{Z}^+$, let T_n be a subset which is maximal with respect to the property that the distance between any two elements is at least $\frac{1}{n}$. (Such subsets exist by Zorn's Lemma.) The set T_n can have no limit points, so (because sequential compactness implies limit point compactness) it must be finite. Since every point of X lies at a distance at most $\frac{1}{n}$ from some element of T_n, the set $\bigcup_n T_n$ is a countable dense subset. By Proposition 4.6 this implies that every open cover has a countable subcover. But since sequential compactness implies countable compactness, this countable subcover in turn has a finite subcover, so altogether we have shown that X is quasi-compact. □

4.3. **Products of quasi-compact spaces.**

Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Recall that the product topology on the Cartesian product $X = \prod_i X_i$ is the topology whose subbase is the collection of all sets of the form $\pi_i^{-1}(U_i)$, where $\pi_i : X \to X_i$ is projection onto the ith factor and U_i is an open set in X_i.

An easy and important fact:

Theorem 4.8. *Let $\mathbf{x} : J \to \prod_i X_i$ be a net in $X = \prod_i X_i$. TFAE:*
a) The net $\mathbf{x}$ converges to $x = (x_i)$ in X.
b) For all i, the image net $\pi_i(\mathbf{x})$ converges to x_i in X_i.

Proof. Continuous functions preserve net convergence, so a) $\implies$ b). Conversely, suppose that $\mathbf{x}$ does not converge to x. Then there exists a finite subset $\{i_1, \ldots, i_n\}$ of I and open subsets U_{i_k} of x_{i_k} in X_{i_k} such that $\mathbf{x}$ is not eventually in $\cap_{k=1}^n \pi_{i_k}^{-1}(U_{i_k})$, which in fact means that for some k $\mathbf{x}$ is not eventually in $\pi_{i_k}^{-1}(U_{i_k})$. But then $\pi_{i_k}(\mathbf{x})$ is not eventually in U_{i_k} and hence does not converge to x_{i_k}. $\square$

We can now prove one of the truly fundamental theorems in general topology.

Theorem 4.9. *(Tychonoff) For a family $\{X_i\}_{i\in I}$ of nonempty spaces, TFAE:*
a) Each factor space X_i is quasi-compact.
b) The Cartesian product $X = \prod_{i\in I} X_i$ is quasi-compact in the product topology.

Proof. That b) implies a) follows from Exercise 4.1.1, since X_i is the image of X under the projection map X_i. Conversely, assume that each factor space X_i is quasi-compact. To show that X is quasi-compact, we shall use the notion of universal nets: by Theorem 4.1 it suffices to show that every universal net $\mathbf{x}$ on X is convergent. But since $\mathbf{x}$ is universal, by Exercise 3.3.2 each projected net $\pi_i(\mathbf{x})$ is universal on X_i. Since X_i is quasi-compact, Theorem 4.1 implies that $\pi_i(\mathbf{x})$ converges, say to x_i. But then by Theorem 4.8, $\mathbf{x}$ converges to $x = (x_i)$: done! $\square$

This proof is due to J.L. Kelley [Ke50a]. To my knowledge, it remains the outstanding application of universal nets.

Exercise 4.3.1 (Little Tychonoff): Let $\mathbf{x}_n$ be a sequence of metrizable spaces. Prove the Tychonoff theorem in this case by combining the following observations –
(i) A countable product of metrizable spaces is metrizable.
(ii) Sequential compactness is equivalent to quasi-compactness in metrizable spaces.
(iii) A sequence converges in a product space iff each projection converges –
with a diagonalization argument. In particular, deduce the Heine-Borel theorem in $\mathbb{R}^n$ from the Heine-Borel theorem in $\mathbb{R}$.

Exercise 4.3.2: Investigate to what extent the Axiom of Choice (AC) is used in the proof of Tychonoff's theorem. Some remarks:
a) The use of Zorn's Lemma in the proof that every net has a universal subnet is unavoidable in the sense that this assertion is known to be equivalent to the **Boolean Prime Ideal Theorem** (BPIT). BPIT is known to require AC (in the sense of being unprovable from Zermelo-Frankel set theory) but not to imply it.

b) A cursory look at the proof might suggest that BPIT implies Tychonoff's theorem. However, Kelley showed [Ke50b] that Tychonoff's theorem *implies* AC,[10] so AC must get invoked again in the proof of Tychonoff. Where? (Hint: BPIT implies that products of quasi-compact Hausdorff spaces are quasi-compact Hausdorff!)

5. Filters

5.1. **Filters and ultrafilters on a set.** Let X be a set. A **filter** on X is a nonempty family $\mathcal{F}$ of nonempty subsets of X satisfying

(F1) $A_1,\ A_2 \in \mathcal{F} \implies A_1 \cap A_2 \in \mathcal{F}$.
(F2) $A_1 \in \mathcal{F},\ A_2 \supset A_1 \implies A_2 \in \mathcal{F}$.

Example 5.1.1: For any nonempty subset Y of X, the collection $\mathcal{F}_Y = \{A \mid Y \subset A\}$ of all subsets containing Y is a filter on X. Such filters are said to be **principal**.

Exercise 5.1.2: Show that every filter on a finite set is principal. (Hint: if $\mathcal{F}$ is a filter on the finite set X then $\cap_{A \in \mathcal{F}} A \in \mathcal{F}$.)

Example 5.1.3: For any infinite set X, the family of all cofinite subsets of X is a filter on X, called the **Fréchet filter**.

Exercise 5.1.4: A filter $\mathcal{F}$ on X is **free** if $\cap_{A \in \mathcal{F}} A = \emptyset$.
a) Show that a principal filter is not free.
b) Show that a filter is free iff it contains the Fréchet filter.

Example 5.1.5: If X is a topological space and $x \in X$, then the collection $\mathcal{N}_x$ of neighborhoods of x is a (nonfree) filter on X. It is principal iff x is an isolated point of X. More generally, if Y is a subset of X, then the collection $\mathcal{N}_Y$ of neighborhoods of Y (recall that we say that N is a neighborhood of Y is $Y \subset N^\circ$) is a nonfree filter on X, which is principal iff Y is an open subset.

Exercise 5.1.6:
a) Let $\{\mathcal{F}_i\}_{i \in I}$ be an indexed family of filters on a set X. Show that $\cap_{i \in I} \mathcal{F}_i$ is a filter on X, the largest filter which is contained in each $\mathcal{F}_i$.
b) Let X be a set with cardinality at least 2. Exhibit filters $\mathcal{F}_1$, $\mathcal{F}_2$ on X such that there is no filter containing both $\mathcal{F}_1$ and $\mathcal{F}_2$.

The collection of all filters on a set X is partially ordered under containment. Exercise 5.1.6a) shows that in this poset arbitrary *joins* exist – i.e., any collection of filters admits a greatest lower bound – whereas Exercise 5.1.6b) shows that if $\#X > 1$ the collection of filters on X is not directed. If $\mathcal{F}_1 \subset \mathcal{F}_2$ we say $\mathcal{F}_2$ **refines** $\mathcal{F}_1$, or is a **finer filter** than $\mathcal{F}_1$. An **ultrafilter** on X is a maximal element with respect to this ordering, i.e., a filter which is not properly contained in any other filter.

Exercise 5.1.7: Let Y be a nonempty subset of X. Then the principal filter $\mathcal{F}_Y$ is

[10] It is sometimes said that this is not surprising, since without AC the Cartesian product might be empty. But I have never understood this remark, since the empty set is of course quasi-compact. At any rate, the proof is not trivial.

an ultrafilter iff $\#Y = 1$.

If X is finite, this gives all the ultrafilters on X. More precisely, the ultrafilters on a finite set may naturally be identified with the elements x of X. However, if X is infinite there are a great many nonprincipal ultrafilters.

Lemma 5.1. *Any filter is contained in an ultrafilter.*

Proof. Since the union of a chain of filters is itself a filter, this follows immediately from Zorn's Lemma. $\square$

Proposition 5.2. *For a filter $\mathcal{F}$ on X, the following are equivalent:*
(i) For every subset Y of X, $\mathcal{F}$ contains exactly one of Y and $X \setminus Y$.
(ii) $\mathcal{F}$ is an ultrafilter.

Proof. If a filter $\mathcal{F}$ satisfies (i) and Y is any subset of X which is not an element of $\mathcal{F}$, then $X \setminus Y \in \mathcal{F}$, and since any finer filter $\mathcal{F}'$ would contain $X \setminus Y$, by (F1) it certainly cannot contain Y; i.e., $\mathcal{F}$ is not contained in any finer filter. Conversely, suppose that $\mathcal{F}$ is an ultrafilter and Y is a subset of X. Suppose first that for every $A \in \mathcal{F}$ we have $A \cap Y \neq \emptyset$. Then the family $\mathcal{F}'$ of all sets containing a set $A \cap Y$ with $A \in \mathcal{F}$ is easily seen to be a filter containing $\mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter we have $\mathcal{F}' = \mathcal{F}$ and in particular $Y = Y \cap X \in \mathcal{F}$. Otherwise there exists an $A \in \mathcal{F}$ such that $A \cap Y = \emptyset$. Then $A \subset X \setminus Y$ and by (F2) $X \setminus Y \in \mathcal{F}$. $\square$

Corollary 5.3. *A nonprincipal ultrafilter is free.*

Proof. If there exists $x \in \bigcap_{A \in \mathcal{F}} A$, then $X \setminus \{x\}$ is not an element of $\mathcal{F}$, so by Proposition 5.2 $\{x\} \in \mathcal{F}$ and $\mathcal{F} = \mathcal{F}_{\{x\}}$.[11] $\square$

In particular free ultrafilters exist on any infinite set: by Lemma 5.1 the Fréchet filter is contained in some ultrafilter, and any refinement of a free filter is free. To be sure, a free ultrafilter is a piece of set-theoretic devilry: it has the impressively decisive ability to, given any subset Y of X, select exactly one of Y and its complement $X \setminus Y$. A bit of thought suggests that even on $X = \mathbb{Z}^+$ this will be difficult or impossible to do in any constructive way. And indeed Lemma 5.1 is known to be equivalent to the Boolean Prime Ideal Theorem, so that it requires (but is not equivalent to) the Axiom of Choice.

Theorem 5.4. *There are $2^{2^{\#X}}$ nonprincipal ultrafilters on an infinite set X.*

For the proof, search for "number of ultrafilters" at `http://www.planetmath.org`.

Exercise 5.1.8: Every filter is the intersection of the ultrafilters containing it.

Exercise 5.1.9: For a nonempty set X, let βX be the set of ultrafilters on X. For $Y \subset X$, let $U(Y) = \{\mathcal{F} \in \beta X \mid Y \in \mathcal{F}\}$.
a) Show that the $U(Y)$ form a base for a compact topology on βX.
b) Show that the map $\beta : X \to \beta(X),\ x \mapsto \mathcal{F}_x$ is an embedding with dense image.

[11] Henceforth we will write $\mathcal{F}_x$ in place of $\mathcal{F}_{\{x\}}$.

5.2. Prefilters.

Proposition 5.5. *For a family F of nonempty subsets of a set X, TFAE:*
(i) For all $A_1,\ A_2 \in F$, there exists $A_3 \in F$ such that $A_3 \subset A_1 \cap A_2$.
(ii) The collection of all subsets which contain some element of F is a filter.

Exercise 5.2.1: Prove Proposition 5.5.

We shall call a family F of nonempty subsets satisfying (i) a **prefilter**.[12] The collection $\mathcal{F}$ of all supersets of F is called the **filter generated by** F (or sometimes the **associated filter**). Note that the situation is reminiscent of the criterion for a family of subsets to be the base for a topology.

Example 5.2.2: Let X be a set and $x \in X$. Then $F = \{\{x\}\}$ is a prefilter on X (which might justifiably be called "constant"). The filter it generates is the principal ultrafilter $\mathcal{F}_x$.

Example 5.2.3: Let X be a topological space and Y a subset of X. Then the collection N_Y of all open neighborhoods of Y (i.e., open sets containing Y) is a prefilter, whose associated filter is the neighborhood filter $\mathcal{N}_Y$ of Y.

Our choice of terminology "prefilter" rather than "filter base" is motivated by the following principle: if we have in mind a certain property P of filters and we are seeking an analogous property for prefilters, then we need merely to define a prefilter to have property P if the filter it generates has property P. Then, if necessary, we unpack this definition more explicitly.

For instance, we can use this philosophy to endow the collection of prefilters on X with a quasi-ordering: we say that a prefilter F_2 **refines** F_1 and write $F_1 \leq F_2$ if for the corresponding filters $\mathcal{F}_1$ and $\mathcal{F}_2$ we have $\mathcal{F}_1 \subset \mathcal{F}_2$. It is not hard to see that this holds iff for every $A_1 \in F_1$ there exists $A_2 \in F_2$ such that $A_1 \supset A_2$. If $F_1 \leq F_2 \leq F_1$ we say that F_1 and F_2 are **equivalent** prefilters and write $F_1 \sim F_2$.

Exercise 5.2.4: If $\#X \geq 2$, show that there exist prefilters F_1 and F_2 on X such that $F_1 \sim F_2$ but $F_1 \neq F_2$.

Similarly we say a prefilter F on X is **ultra** if its associated filter is an ultrafilter. This amounts to saying that for any $Y \subset X$, there exists $A \in F$ such that either $A \subset Y$ or $A \subset (X \setminus Y)$.

Exercise 5.2.5 (Filter subbases):
a) Show that for a family I of nonempty subsets of a set X, TFAE:
(i) I has the finite intersection property: if $A_1, \ldots, A_n \in I$, then $A_1 \cap \ldots A_n \neq \emptyset$.
(ii) There exists a prefilter F such that $I \subset F$.
(iii) There exists a filter $\mathcal{F}$ such that $I \subset \mathcal{F}$.
b) If I satisfies the equivalent conditions of part a), show that there is a unique minimal filter $\mathcal{F}(I)$ containing I, called the **filter generated by** I.

[12] The more traditional terminology is **filter base**.

A family $\{F_i\}_{i\in I}$ of prefilters on a set X is **compatible** if there exists a prefilter $F \supset \bigcup_{i\in I} F_i$, i.e., if $\bigcup_{i\in I} F_i$ is a filter subbase. (It is equivalent to require that $\bigcup_{i\in I} F_i$ be refined by some prefilter.) In turn, this occurs iff for every finite subset $J \subset I$ and any assignment $j \mapsto A_j \in F_j$ we have $\bigcap_{j\in J} A_j \neq \emptyset$.

5.3. **Convergence via filters.**

Let F be a prefilter in a topological space X, and let x be a point of X. We say that F **converges** to x – and write $F \to x$ – if F *refines* the neighborhood filter $\mathcal{N}_x$ of x. In plainer language, this means that every neighborhood N of x contains an element A of F.

Let F be a prefilter in a topological space X, and let x be a point of X. We say that x is a **limit point**[13] of F if F *is compatible with* the neighborhood filter $\mathcal{N}_x$, or in plainer language, if every element of F meets every neighborhood of x.

Proposition 5.6. *Let F be a prefilter on X with associated filter $\mathcal{F}$, and let $F' \geq F$ be a finer prefilter.*
a) If F converges to x, then x is a limit point of F.
b) F converges to $x \iff \mathcal{F}$ converges to x.
c) x is a limit point of $F \iff x$ is a limit point of $\mathcal{F}$.
d) If F converges to x, then F' converges to x.
e) If x is a limit point of F', then x is a limit point of F.
f) X is Hausdorff $\iff$ every prefilter on X converges to at most one point.

Exercise 5.3.1: Prove Proposition 5.6.

Proposition 5.7. *Let F be a prefilter on X. TFAE:*
(i) x is a limit point of F.
(ii) There exists a refinement F' of F such that F' converges to x.

Proof. (i) $\implies$ (ii): If x is a limit point of F, there exists a prefilter F' refining both F and $\mathcal{N}_x$, and then F' is a finer prefilter converging to x.
(ii) $\implies$ (i): since $F' \to x$, x is a limit point of F' (Proposition 5.6a)), and since $F' \geq F$, x is a limit point of F (Proposition 5.6e)). □

Proposition 5.8. *Let X be a topological space, Y a nonempty subset of X and x a point of x. TFAE:*
(i) x is a limit point of the prefilter $F_Y = \{Y\}$.
(ii) $x \in \overline{Y}$.

Proof. Both (i) and (ii) amount to: every neighborhood of x meets Y. □

A more traditional characterization of closure using filters is the following:

Corollary 5.9. *Let X be a topological space, Y a nonempty subset of X and x a point of x. TFAE:*
(i) There exists a filter $\mathcal{F}$ such that $Y \in \mathcal{F}$ and $\mathcal{F} \to x$.
(ii) $x \in \overline{Y}$.

[13] Alternate terminology: **cluster point**

Proof. If (i) holds, then for every neighborhood N of x we have $N \in \mathcal{F}$ and $Y \in \mathcal{F}$, hence $N \cap Y$ is in $\mathcal{F}$ and thus nonempty. If (ii) holds, then Proposition 5.8 supplies us with a prefilter F (namely F_Y) which has x as a limit point. Applying Proposition 5.7 we get a prefilter F' which converges to x and, being a refinement of F, contains some subset Z of Y. Then the filter generated by F' converges to x and now must contain Y as an element. □

Proposition 5.10. *Let X be a topological space, Y a nonempty subset of X and x a point of x. TFAE:*
(i) The prefilter $F_Y = \{Y\}$ is compatible with the neighborhood filter $\mathcal{N}_x$ of x.
(ii) $x \in \overline{Y}$.

Proof. Each of (i) and (ii) says that every neighborhood of x meets Y. □

Lemma 5.11. *If an ultra prefilter F has x as a limit point, then $F \to x$.*

Proof. As above, there exists a prefilter F' which refines both F and $\mathcal{N}_x$. But since F is ultra, it is equivalent to all of its refinements, so that F itself refines $\mathcal{N}_x$. □

It may not come as a surprise that we can get further characterizations of quasicompactness in terms of convergence / limit points of prefilters.

Theorem 5.12. *For a topological space X, TFAE:*
(i) X satisfies the equivalent conditions of Theorem 4.1 ("X is quasicompact.")
(ii) Every prefilter on X has a limit point.
(iii) Every ultra prefilter on X is convergent.
The same equivalences hold with "'prefilter" replaced by "filter" in (ii) and (iii).

Proof. (i) $\implies$ (ii): Let $F = \{A_i\}$ be a prefilter on X. The sets A_i satisfy the finite intersection property, hence *a fortiori* so do their closures. Appealing to condition e) in Theorem 4.1 there is an $x \in \bigcap_i \overline{A_i}$, and this means precisely that each A_i meets each neighborhood of x.
(ii) $\implies$ (iii) follows immediately from Lemma 5.11.
(iii) $\implies$ (i): Consider a family $I = \{F_i\}$ of closed subsets of X satisfying the finite intersection condition. Then I is a filter subbase, so that there exists some ultra prefilter refining I. By hypothesis, there exists $x \in X$ such that F converges to x, and *a fortiori* x is a limit point of F. So every element of F – and in particular each F_i – meets every neighborhood of x, so that $x \in \overline{F_i} = F_i$. Therefore $\cap_i F_i$ contains x and is thus nonempty. □

The fact that the results hold also for filters instead of prefilters is left to the reader.

Pushing forward filters: if $f : X \to Y$ is any map of sets and $I = \mathcal{A}_\rangle$ is a family of subsets of X, then by $f(I)$ we mean the family $\{f(A_i)\}_{i \in I}$.

Proposition 5.13. *Let $f : X \to Y$ be a function and F a prefilter on X.*
a) $f(F)$ is a prefilter on Y.
b) If F is ultra, so is $f(F)$.

Exercise 5.3.2: Prove Proposition 5.13.

Proposition 5.14. *Let $f : X \to Y$ be a function. TFAE:*
(i) For every prefilter F on X with a limit point x, $f(F)$ has $f(x)$ as a limit point.
(ii) For every prefilter F on X converging to x, $f(F)$ converges to $f(x)$.
(iii) f is continuous.

Proof. A function f between topological spaces is continuous iff for all $x \in X$, $f(\mathcal{N}_x)$ is a neighborhood base for Y. The result follows easily from this and is left to the reader. $\square$

Definition: Let $\{X_i\}_{i\in I}$ be an indexed family of topological spaces and suppose given a prefilter F_i on each X_i. We then define the **product prefilter** $\prod_i F_i$ to be the family of subsets of X of the form $\prod_{i\in I} M_i$, where there exists a finite subset $J \subset I$ such that $M_i = X_i$ for all $i \in I \setminus J$ and $M_i \in F_i$ for all $i \in J$. Since

$$(\prod_{i\in I} M_i) \cap (\prod_{i\in I} M_i') = \prod_{i\in I}(M_i \cap M_i') \supset \prod_{i\in I} M_i''$$

where M_i'' is an element of F_i contained in $M_i' \cap M_i''$ (or is X_i if $M_i = M_i'' = X_i$), this does indeed give a prefilter on X. Another way around is to say that F is the prefilter generated by taking finite intersections of the filter subbase $\pi_i^{-1}(M_i)$.

Exercise 5.3.3: a) If for each i we are given equivalent prefilters $F_i \sim F_i'$ on X_i, then the product prefilter $\prod_i F_i$ is equivalent to $\prod_i F_i'$.
b) (Remark): Because of part a), as far as convergence / limit points are concerned, it would be no loss of generality to assume that $X_i \in F_i$ for all i, and then we get a cleaner definition of the product prefilter.

Theorem 5.15. *Let F be a prefilter on the product space $X = X_i$. TFAE:*
(i) F converges to $x = (x_i)$.
(ii) For all i, $\pi_i(F)$ converges to x_i.

Proof. (i) $\implies$ (ii) is immediate from Proposition 5.14, so assume (ii). It is enough to show that for every $i \in I$ and every neighborhood N_{ij} of x_i in X_i there exists an element $A \in F$ with $\pi_i(A) \subset N_{ij}$, for then F will be a prefilter which is finer than the family $\pi_i^{-1}(N_{ij})$ which is a subbasis for the filter of neighborhoods of x in X. But this is tautological: since $\pi_i(F)$ converges to x_i, it contains an element, say $B = \pi_i(A)$, which is contained in N_{ij}, and then $A \subset \pi_i^{-1}(N_{ij})$. $\square$

Now for a proof of Tychonoff's Theorem (Theorem 4.9) using filters:

That b) implies a) follows from Exercise 4.1.1, since X_i is the image of X under the projection map X_i. Conversely, assume that each factor space X_i is quasi-compact. To show that X is quasi-compact, we shall use the notion of ultra prefilters: by Theorem 5.12 it suffices to show that every ultra prefilter F on X is convergent. Since F is ultra, by Proposition 5.13b) each projected prefilter $\pi_i(F)$ is ultra on X_i. Since X_i is quasi-compact, Theorem 5.12 implies that $\pi_i(F)$ converges, say to x_i. But then by Theorem 5.15, F converges to $x = (x_i)$: done!

This proof is due to H. Cartan [Ca37b].

6. The correspondence between filters and nets

Take a moment and compare Cartan's ultra prefilter proof with Kelley's universal net proof. By replacing every instance of "universal net" with "ultra prefilter" they become word for word identical! This, together with the other manifest parallelisms between §3 and §5, strongly suggests that nets and prefilters are not just different means to the same end but are somehow directly related: given a net, there ought to be a way to trade it in for a prefilter, and vice versa, in such a way as to preserve the concepts of: convergence, limit point, subnet / finer prefilter and universal net / ultra prefilter. This is exactly the correspondence that we now pursue.

If we search the preceding material for hints of how to pass from a net to a prefilter, sooner or later we will notice that we have already done so in the proof that b) $\implies$ e) in Theorem 4.1. We repeat that construction here, after introducing the following useful piece of notation.

If $\leq$ is a relation on a set I, for $i \in I$ we put $i^+ = \{i' \in I \mid i \leq i'\}$.

Proposition 6.1. *Let $\mathbf{x} : I \to X$ be a net in the set X. Then the collection $\mathcal{P}(\mathbf{x}) := \{i^+\}_{i \in I}$ is a prefilter on X, the* ***prefilter of tails*** *of $\mathbf{x}$.*

Proof. Indeed, for $i_1,\ i_2 \in I$, choose $i_3 \geq i_1,\ i_2$. Then $A_{i_3} \subset A_{i_1} \cap A_{i_2}$. ☐

Conversely, suppose we are given a prefilter F on X: how to get a net? The first (and usually harder) task is to find the directed index set I and the second is to define the mapping $I \to X$. The key observation is that the condition $A_1,\ A_2 \in F \implies \exists A_3 \in F \mid A_3 \subset A_1 \cap A_2$ on a nonempty family of nonempty subsets of X says precisely that the elements of F are (like the neighborhoods of a point) directed under reverse inclusion. This suggests that we should take $I = F$. Then to get a net we are supposed to choose, for each $A \in F$, some element x_A of X. Other than to require $x_A \in A$, no condition presents itself. Making many arbitrary choices is dismaying, on the one hand for set-theoretic reasons but moreover because we shall inevitably have to worry about whether our choices are correct. So let's worry: once we have our net $\mathbf{x}(F)$, we can apply the previous construction to get another prefilter $\mathcal{P}(\mathbf{x}(F))$, and whether we dare to admit it out loud or not, we are clearly hoping that $\mathcal{P}(\mathbf{x}(F)) = F$.

Let us try our luck on the simplest possible example: let X be a set with more than one element, and let $F = \{X\}$, the unique *minimal* filter. A net $\mathbf{x}$ with index set F is just a choice of a point $x \in X$. The corresponding prefilter $\mathcal{P}(\mathbf{x})$ – namely the principal prefilter $F_x = \{x\}$ – is not only not equal to F, it is *ultra*: its associated filter is *maximal.* At least we don't have to worry about our choice of x_A in A: all choices fail equally.

We trust that we have now suitably motivated the correct construction:

Proposition 6.2. *Let F be a prefilter on X. Let $I(F)$ be the set of all pairs (x, A) such that $x \in A \in F$. We endow $I(F)$ with the relation $(x_1, A_1) \leq (x_2, A_2)$ iff $A_1 \supset A_2$. Then $(I(F), \leq)$ is a directed set, and the assignment $(x, A) \mapsto x$ defines a net $x(F) : I \to X$.*

Exercise 6.1: Prove Proposition 6.2.

Coming back to our earlier example, if $F = \{X\}$, then $\mathbf{x}(F)$ has domain $I = \{X\} \times X$ and is just $(X, x) \mapsto x$. Note that the induced quasi-ordering on X makes $x \leq x'$ for any x, x': notice that it *is* directed and *is not* anti-symmetric (which at last justifies our willingness to entertain directed *quasi*-ordered sets). So for any $x \in X$, we have $(X, x)^+ = \{x' \geq x\} = X$, and we indeed get $\mathcal{P}(\mathbf{x}(F)) = \{X\} = F$. This was not an accident:

Proposition 6.3. *For any prefilter G on X, we have $F(x(G)) = G$.*

Proof. The index set of $x(G)$ consists of all pairs (x, A) for $x \in A \in F$, partially ordered under reverse inclusion. The associated prefilter consists of sets $A_{(x,A)} = \{\pi_1((x', A')) \,|(x', A') \geq (x, A)\}$. A moment's thought reveals this to be the set of all points x in filter elements $A' \subset A$, i.e., $A_{(x,A)} = A$. □

What about the relation $x(F(\mathbf{x})) = \mathbf{x}$? A moment's thought shows that this cannot possibly hold in general: the index set I of any net associated to a prefilter on X is a subset of $X \times 2^X$ hence has cardinality at most $\#(X \times 2^X)$ (i.e., $2^{\#X}$ is X is infinite), but every nonempty set admits nets based on index sets of arbitrarily large cardinality, e.g. constant nets. Indeed, if $\mathbf{x} : I \to X$ has constant value $x \in X$, then the associated prefilter $F(\mathbf{x})$ is just $\{x\}$, and then the associated net $x(F(\mathbf{x}))$ has $I = \{(x, \{x\})\}$, a one point set!

Exercise: Suppose that a net $\mathbf{x}$ is eventually constant, with eventual value $x \in X$.
a) Show that the filter generated by $F(\mathbf{x})$ is the principal ultrafilter $\mathcal{F}_x$.
b) Suppose that F is a prefilter generating the principal ultrafilter $\mathcal{F}_x$
(i.e., $\{x\} \in F$!). Show that $\mathbf{x}(F)$ is eventually constant with eventual value x.

Nevertheless the nets $\mathbf{x}$ and $x(F(\mathbf{x}))$ are "pan-topologically equivalent" in the sense that they converge to the same points and have the same limit points for any topology on X. Indeed:

Theorem 6.4. *Let X be a topological space, F be a prefilter on X, $\mathbf{x}$ a net on X and $x \in X$.*
a) F converges to $x \iff x(F)$ converges to x.
b) $\mathbf{x}$ converges to $x \iff F(\mathbf{x})$ converges to x.
c) x is a limit point of $F \iff x$ is a limit point of $x(F)$.
d) x is a limit point of $\mathbf{x} \iff x$ is a limit point of $F(\mathbf{x})$.
e) F is an ultra prefilter $\iff x(F)$ is a universal net.
f) $\mathbf{x}$ is a universal net $\iff F(\mathbf{x})$ is an ultra prefilter.
g) If $\mathbf{y}$ is a subnet of $\mathbf{x}$, then $F(\mathbf{y})$ refines $F(\mathbf{x})$.

Exercise 6.2: Prove Theorem 6.4

Were you expecting a part h)? Unfortunately it need not be the case that if $F' \geq F$ then the associated net $x(F')$ can be endowed with the structure of a subnet of $x(F)$. A bit of quiet contemplation reveals that a subnet structure is equivalent to the existence of a function $r : F' \to F$ satisfying $A' \subset r(A')$ for all $A' \in x(F')$ and $A'' \subset A' \implies r(A'') \subset r(A')$. To see that such a map need not exist, take $X = \mathbb{Z}^+$. For all $n \in \mathbb{Z}^+$, define let $A_n = \{1\} \cup \{n, n+1, \ldots\}$. Since $A_n \cap A_m = A_{\max m,n}$, $F = \{A_n\}$ is a prefilter on X. Let $F' = F \cup \{1\}$. The directed set I' on which $x(F')$ is based has an element which is larger than every element – namely $\{(1, \{1\}\}$ – but this does not hold for the directed set I on which $x(F)$ is based. (Indeed, I is order isomorphic to the positive integers, or the ordinal ω, whereas I' is order isomorphic to $\omega + 1$.) There is therefore no order homomorphism $I' \to I$ so that $x(F')$ cannot be given the structure of a subnet of $x(F)$.

This example isolates the awkwardness of the notion of subnet. Taking a step back, we see that we became satisfied that we had the right definition of a subnet only insofar as it fit in to the theory of convergence as it shoud: i.e., it rendered true the facts that "x is a limit point of $\mathbf{x} \iff$ some subnet $\mathbf{y}$ converges to x" and "every net $\mathbf{x}$ admits a subnet $\mathbf{y}$ which converges to each of its limit points." These two results are what subnets *are for*. Now that we have at our disposal the correspondence with the theory of filters, the extent of our leeway becomes clear: any definition of "$\mathbf{y}$ is a subnet of $\mathbf{x}$" which satisfies the following requirements:

(SN1) If $\mathbf{y}$ is a subnet of $\mathbf{x}$, then $F(\mathbf{y}) \geq F(\mathbf{x})$;
(SN2) For every net $\mathbf{x} : I \to X$ and every prefilter $F' \geq F(\mathbf{x})$, there exists a subnet $\mathbf{y}$ of $\mathbf{x}$ with $F(\mathbf{y}) = F'$;

will verify the above results, and hence serve as a definition. Note that (SN1) is part g) of Theorem 6.4. The following establishes (SN2) (and a little more).

Theorem 6.5. *(Smiley) Let $\alpha : I \to X$ be a net, and let F' be a prefilter on X which is compatible with $F(\mathbf{x})$. Let $\mathcal{I}$ be the set of all triples (x, i, A) with $i \in I$, $A \in F'$ and $x \in A$ such that there exists $j \geq i$ with $\alpha_j = x$. Let $\leq$ be the relation on $\mathcal{I}$ by $(x, i, A) \leq (x', i', A')$ if $i \leq i'$ and $A \supset A'$. Let $\gamma : \mathcal{I} \to X$ be the function $(x, i, A) \mapsto x$. Then:*
a) $\mathcal{I}$ is a directed set, and γ is a net on X.
b) Via the natural map $\mathcal{I} \to I$ given by $(x, i, A) \mapsto i$, γ is a subnet of I.
c) The associated prefilter $F(\gamma)$ is the prefilter generated by $F(\mathbf{x})$ and F'.
So if $F' \geq F(\mathbf{x})$, then γ is a subnet of $\mathbf{x}$ with $F(\gamma) = F'$.

Exercise 6.3: Prove Theorem 6.5.

Thus we have shown convincingly that our definition of subnet is an acceptable one in the sense of (SN1) and (SN2). (In particular, the material of this section and §4 on filters gives independent proofs of the material of §3.

However, from the filter-theoretic perspective there is certainly a simpler definition of subnet that renders true (SN1) and (SN2): just define $\mathbf{y} : J \to X$ to be a subnet of $\mathbf{x} : I \to X$ if $F(\mathbf{y}) \geq F(\mathbf{x})$; or, in other words, that for all $i \in I$, there exists $j \in J$ such that $y(j^+) \subset x(i^+)$. That this should be the definition of a subnet was in fact suggested by Smiley.

7. Notes

The material of §1 ought to be familiar to every undergraduate student of mathematics. Among many references, we can recommend Kaplansky's elegant text [Ka]. That the key properties of metric spaces making the theory of sequential convergence go through are first countability and (to a lesser extent) Hausdorffness was first appreciated by Hausdorff himself. There is a very rich theory of the sequential closure operator, e.g. in set-theoretic topology (via the sequential order) [AF68].

The development of a repaired convergence theory via nets has a complicated history. In some form, the concept was first developed by E.H. Moore in his 1910 colloquium lectures [M10] and then in his 1915 note *Definition of limit in general integral analysis* [Mo15]. A fuller treatment was given in the 1922 paper [MS22], written jointly with his student H.L. Smith. As the titles of these articles suggest, Moore and Smith were primarily interested in analytic applications: as in §3.2, the emphasis of their work was on a single notion of limit to which all the various complicated-looking limiting processes one meets in analysis can refer back to.[14] Thus their theory was limited to "Moore-Smith sequences" (i.e., nets) with values in $\mathbb{R}$, $\mathbb{C}$, or some Banach space.

In 1937, G(arrett) Birkhoff published a paper [Bi37] whose point of departure is the same as ours: to use mappings from a directed set to a topological space to generalize facts about neighborhoods, closure and continuous functions that hold using sequences only under the assumption of first countability. He then goes on to discuss applications to the *completion* of various structures of mixed algebraic/topological character, e.g. topological vector spaces and topological algebras. In this aspect he goes beyond the material we have presented so far and competes with the work of André Weil, who in that same year introduced the seminal concept of **uniform space** [We] as the correct generalization of special classes of spaces, notably metric spaces and topological groups, in which one can speak of one pair of points being as close together as another.

In 1940 Tukey published a short book [T] which explored the interrelationships of Moore-Smith convergence and Weil's uniform spaces. Tukey's book is systematic and foundational, in particular employing a language which does not seem to have persuaded many to speak. (E.g. we find in his book that a **stack** is the directed set of finite subsets of a given set S – if only that's what stack meant today! – and a **phalanx** if a function from a stack to a topological space (cf. Example 3.2.1).) The book is probably most significant for its formulation of the notion of a uniform space in terms of **star refinements**, which is still in use (e.g. [Ba06]). Moreover the notion of uniform completion seems to appear here for the first time. We quote the first two sentences of Steenrod's review of Tukey's book: "The extension of metric methods to non-metrizable topological spaces has been a principal development in topology of the past few years. This has occurred in two directions: one through a rebirth of interest in Moore-Smith convergence due to results of Garrett Birkhoff, and the other through the concept of uniform structure due to André Weil." May it not even be the case that the emerging study of uniform spaces was the major cause of the rebirth of interest in Moore-Smith convergence?

Our treatment of nets in §3 closely follows Kelley's 1950 paper *Convergence in topology* [Ke50a] and his text *General Topology* [Ke]. Apart from introducing the term "nets" for the first time, [Ke50a] is the first to recognize the subnet as an essential tenet of the theory, to prove Proposition 3.5, to introduce the notion of universal net and apply it, and to give a strikingly simple proof of Tychonoff's theorem. On the other hand the idea of a universal net is motivated by that of an ultrafilter, and Kelley makes explicit reference to earlier work of H. Cartan.

Indeed, in 1937 Henri Cartan came up with the definition of a filter: apparently inspiration struck during a lull in a Séminaire Bourbaki (and Cartan stayed behind to think about his new idea rather than go hiking with the rest of the group). His ideas are written up briefly in two *Comptes Rendus* [Ca37a], [Ca37b]. He had no trouble convincing the other Bourbakistes of the importance of this idea: Bourbaki's 1940 text *Topologie Generale* introduces filters early on and uses them systematically

[14] It is therefore a bit strange, is it not, that one does not learn about nets in basic real analysis courses? Admittedly the abstract Lebesgue integral plays a similar unifying role.

throughout. It may well be the case that this was the most influential of the many innovations introduced across Bourbaki's many books.

Bourbaki's treatment of filters is much more extensive than what we have given here. In particular Bourbaki rewrites the theory of convergent series and integrals in the filter-theoretic language. To my taste this becomes tiresome and serves as a *de facto* demonstration of the usefulness of nets in more analytic applications. One Bourbakism we have adopted here is the emphasis of the development of the theory at the level of prefilters (called there and elsewhere "filter bases"). It is not necessary to do so – at any stage, one can just pass to the associated filter – but seems to lead to a more precise development of the theory. We have emphasized the notion of compatible prefilters more than is typical (an exception is [Sm57]). The existence of free ultrafilters even on a countably infinite set leads to what must be the single most striking application of set-theoretic machinery in general mathematics, the **ultraproduct**. The proof of Tychonoff's theorem via ultrafilters first appears in [Bo] and is one of Bourbaki's most celebrated results.

The material of §6 is distressingly absent from most treatments. Most texts choose to present either the results of §3 or the results of §4 but not both, and then give a few exercises on the convergence theory they did not develop. In terms of relating the two theories, it is standard to drop the unhelpful remark "The equivalence of nets and filters is part of the folklore of the subject." Even [Ke] does this, although he gives the construction of a net from a filter and a filter from a net (the latter amounts to taking the associated filter of our prefilter of tails) and asks the reader to show our Proposition 6.3 (for filters). But this result is cited as "grounds for suspicion" that filters and nets are "equivalent", a phrasing which leads the careful reader to wonder whether things do in fact work out as they appear.

Also of interest is R.G. Bartle's paper *Nets and Filters in Topology* [Ba55]. Written at about the same time as [Ke], it aspires to make explicit the equivalence between the two theories. Unfortunately the paper has some defects: the net that Bartle associates with a filter $\mathcal{F}$ is indexed by the elements of $\mathcal{F}$. As discussed in §6, this is inadequate: upon passing to the (pre)filter of tails, one gets a (pre)filter which may be strictly finer than the original one. (The correct definition is given in a footnote, following the suggestion of the referee!) As a result, instead of the equivalences of Theorem 6.4 Bartle gives only one-sided statements of the form "If the filter converges, then the net converges." Moreover, he erroneously claims [Ba55, Prop. 2.5] that given a net $\mathbf{x}$ and a finer prefilter $F' \geq F(\mathbf{x})$, there exists a subnet $\mathbf{y}$ of $\mathbf{x}$ with $F(\mathbf{y}) = F'$. There is an erratum [Ba55e] to [Ba55] which replaces Prop. 2.5 by our (SN2). In between the 1955 paper and its 1963 erratum comes Smiley's 1957 paper [Sm57], whose results we have presented in §6. It is tempting to derive a moral about the dangers of leaving "folklore" unexamined; we will leave this to the interested reader.

References

[AF68] A.V. Arhangel'skii and S.P. Franklin, *Ordinal invariants for topological spaces.* Michigan Math. J. 15 (1968), 313-320; addendum, ibid. 15 1968 506.

[Ba55] R.G. Bartle, *Nets and Filters in Topology*, Am. Math. Monthly, Vol. 62, No. 8 (Oct., 1955), 551-557.

[Ba55e] R.G. Bartle, *A Correction for "Nets and Filters in Topology"*, Am. Math. Monthly, Vol. 70, No. 1. (Jan., 1963), 52-53.

[Ba06] M. Baker, *Uniform structures and Berkovich spaces.* 2006 preprint.

[Bi37] G. Birkhoff, *Moore-Smith convergence in general topology.* Ann. of Math. (2) 38 (1937), 39-56.

[Bo] N. Bourbaki, *Éléments de Mathématique. Live III: Topologie Générale*, Hermann, Paris, 1940, 1942, 1947, 1948, 1949.

[Ca37a] H. Cartan, *Théorie des filtres.* C.R. Acad. Paris 205 (1937), 595-598.

[Ca37b] H. Cartan, *Filtres et ultrafilters.* C.R. Acad. Paris 205 (1937), 777-779.

[Du64] R.M. Dudley, *On sequential convergence.* Trans. Amer. Math. Soc. 112 (1964), 483-507.

[Fr65] S.P. Franklin, *Spaces in which sequences suffice.* Fund. Math. 57 (1965), 107-115.

[Ka] I. Kaplansky, *Set theory and metric spaces.* Allyn and Bacon Series in Advanced Mathematics. Allyn and Bacon, Inc., Boston, Mass., 1972.

[Ke50a] J.L. Kelley, *Convergence in topology.* Duke Math. J. 17 (1950), 277–283.

[Ke50b] J.L. Kelley, *The Tychonoff product theorem implies the axiom of choice.* Fund. Math. 37 (1950), 75-76.

[Ke] J.L. Kelley, *General topology.* D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[M10] E.H. Moore, *On a form of general analysis*, Yale Colloquium Lectures, 1910.

[Mo15] E.H. Moore, *Definition of limit in general integral analysis*, Proc. Nat. Acad. Sci. vol 1 (1915), p. 628.

[MS22] E.H. Moore and H.L. Smith, *A general theory of limits*, Am. Jour. 44 (1922), 102-121.

[Sm57] M.F. Smiley, *Filters and equivalent nets*, Am. Math. Monthly, Vol. 64, No. 5 (May, 1957), 336-338.

[T] J.W. Tukey, *Convergence and Uniformity in Topology.* Annals of Mathematics Studies, no. 2. Princeton University Press, Princeton, N. J., 1940.

[We] A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale.* Act. Sci. Ind. 551, Paris, 1937.

[Wi] S. Willard, *General topology.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1970.