We can’t do everything.

Class time is precious. Calculus instructors have to decide what to salvage and what to jettison. We do some proofs in class—we do *more* than some—but when we discuss the intermediate value theorem (IVT), presenting a proof hardly seems worth the time. Instead we draw a picture of an improbably straight river and tell our students, “This theorem is *obvious*, because clearly you can’t get from one side of the river to the other without getting wet.” We reassure them, moreover, that we will not even *try* to prove the theorem rigorously, because the proof is so difficult that every calculus text we know either quarantines the proof in an appendix or else gives up and just footnotes another source for it. This kind of presentation saves valuable time for other pressing topics; it seems to have been the standard for hundreds of years and probably will be the standard for hundreds more.

One semester, though, this standard approach made me uneasy. Was I setting a bad example for my students by avoiding the proof? Was I missing an opportunity to sharpen their reasoning skills? Was I damaging my credibility by making such a sweeping claim without showing my work? Above all, wasn’t it strange to dismiss the IVT with this bizarre one-two punch—first stating that it *was obvious* and then, paradoxically, insisting that *its proof was far above my students’ heads*?

I was surprised, but oddly reassured, to realize that both ideas are complete myths. The Intermediate Value Theorem is NOT OBVIOUS. Any claim of its obviousness is based on an unfortunate misunderstanding of what the theorem says. Furthermore, Proving the IVT is NOT DIFFICULT. There is no reason calculus students cannot see how it is done. I aim to demonstrate the latter point by presenting a proof here that works nicely in a calculus class.

First, though, the former point. Why shouldn’t students just accept what their intuition tells them? Why does the IVT need to be proved?
The intermediate value theorem is NOT obvious . . .

I have heard it said that the proof of the IVT can be skipped because students have an intuition about the real line that they can “transfer” to the graphs of continuous functions. That would be fine—if only intuition were a reliable source! Intuition, after all, is what tells students that \((x + y)^2 = x^2 + y^2\), that \((fg)'(x) = f'(x)g'(x)\), and that \(\frac{\ln x}{x}\) reduces to some mysterious object called “\(\ln\)” Intuition is what leads them—having examined perhaps a dozen triangles in their lives, each small enough to fit on a notebook page, each with an angle sum that appeared to round off to 180 degrees when measured with a fifty-cent protractor—to declare an angle sum of exactly 180 for every triangle in the universe, no matter how many parsecs long its sides are.

Concerning the real line, we try to help students develop their intuition by telling them that “there are no gaps,” that “nothing is missing,” and that “everything that SHOULD be there IS there.” Frankly, though, I question my own intuition about the real line. I can illustrate it for myself by drawing a line segment in a notebook; but if I view that page under a microscope, it is more gap than substance, and its points are probably all rational. I can picture the real line in the context of high-school Euclidean geometry; but that guarantees the existence of nothing except surds. There are reals that are not surds. There are reals that are not algebraic, reals that are not computable, reals that do not even correspond to paths in any computable binary tree. I doubt that I truly grasp everything that is on the real line, much less what could have been, and it seems less than honest for me to tell my students that “nothing is missing” or to talk about “what SHOULD be there” unless they (and I) have a coherent idea of what could have been there in the first place.

If I encourage students to accept, unquestioning, whatever their intuition tells them, then I risk teaching them to avoid working on careful arguments or thinking beyond their own experience. (There are more things in heaven and earth, after all, than are dreamt of in a college freshman’s philosophy.) So I am leery of substituting intuition for proof in any situation. This is one reason, the pedagogical one, that I refuse to call the IVT obvious.

The other reason, the mathematical one, is more subtle—and, I believe, even more important. It is that the IVT is almost always misread, and thus misunderstood, by students, by their instructors, by everyone:

**The intermediate value theorem.** If \(f\) is continuous on \([a, b]\), and \(p\) is any real number such that \(f(a) < p < f(b)\) or \(f(b) < p < f(a)\), then there is some \(c\) in the interval \((a, b)\) such that \(f(c) = p\).

When we claim that this is obvious, we are forgetting that the word *continuous* has two meanings. There is the everyday English meaning, the informal and intuitive one, no gaps, no breaks, the pencil not being lifted from the page—and for this sense of the word, let’s use an informal-looking font: *continuous*. The other meaning gets an official-looking, no-nonsense, Germanic font: *continuous*. This is the precise mathematical meaning—i.e., a function is *continuous* at the number \(a\) if and only if for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that, for all \(x\) such that \(|x - a| < \delta\), we have \(|f(x) - f(a)| < \varepsilon\).

If the IVT were a statement about *continuous* functions, then it might seem obvious, but it also would be a mismatching of concepts—ultimately nonmathematical, therefore utterly unhelpful. What functions could we possibly apply it to? I cannot say with certainty that even the squaring function is *continuous*! Over the years I’ve drawn a great many things without lifting my pencil from the page, but I know that none of them was actually the graph of the squaring function, if for no other reason
than that the width of my pencil point necessarily introduced some inaccuracy. (On the
other hand, if the criterion is that I can draw an inaccurate picture without my pencil
leaving the page, then I’m pretty sure that all functions are continuous.)

In any event, the IVT is not about functions. It is about continuous functions. The concepts are not the same: We define continuity to model continuity. We hope that by defining the one we have captured the noteworthy features of the other, but we should not be fooled into assuming that we have succeeded, just because the names match, as if the simple act of naming carried mathematical significance. The IVT is really a check that our definition of continuity is good. To interpret it any other way is to miss the point and undermine the theorem.

I am not the first to point out that there are two distinct ideas at work here. Coffa [1, p. 684], for instance, notes the important difference between being “true in virtue of concepts” and being “true in virtue of definitions,” arguing that Kant confused the two and that Bernard Bolzano, who rigorously proved a special case of the IVT in 1817, was “the first to recognize the fallacy” involved in Kant’s view. Bolzano, in fact, opened his own historic paper by exposing spurious proofs of the IVT, and the first two were derived from errors of exactly the continuous-versus-continuous variety [3, pp. 160–162]. He noted in particular that the ideas of time and motion, which are invoked when students hear that the IVT is about “getting from one side of the river to the other,” have no place in a proof of a general theorem about functions that might not have anything to do with either time or motion [3, p. 161].

To help resolve the continuous/continuous confusion, I propose a slight restatement of the intermediate value theorem:

The intermediate value theorem (clarified). The mathematical definition of continuity captures an important aspect of the informal concept of continuity, to wit, if \( f \) is continuous on \([a, b]\), and \( p \) is any number such that \( f(a) < p < f(b) \) or \( f(b) < p < f(a) \), then there is some \( c \) in the interval \((a, b)\) such that \( f(c) = p \).

That this conclusion follows obviously from the hypothesis of continuity, with all its \( \varepsilon \)'s and \( \delta \)'s and absolute value bars, no one can claim with a straight face. Apparently, then, we do need to prove the IVT, not just accept it blindly! Along the way, we would do well to correct students’ false intuition about the real line—by helping them discover what a real number really is.

Keeping it real

Students know that natural numbers are the counting numbers and zero, our means of expressing quantities of things in front of us. They know that the integers are formed from the natural numbers when we try subtracting the natural numbers from each other, and that the rational numbers are formed from the integers via division. But do they know where real numbers come from?

They should learn, because the concept of real number is the most fundamental notion in calculus, more so than derivatives or integrals or even limits. Most students reach calculus believing only that a real number is “a point on the real number line,” an idea that, for reasons already given, seems inadequate. This is why I explore the matter with my calculus students during our very first week of class. The real numbers must be spawned from the rationals somehow. The completeness axiom makes it possible:

The completeness axiom. Every sequence of rational numbers with a [rational] ceiling has a PERFECT ceiling.
Here *ceiling* and *perfect ceiling* are euphemisms for the potentially frightening terms “upper bound” and “least upper bound.” Once we acknowledge that this is an assumption, and a heavy one at that, we get to the matter at hand:

**Definition.** The *real numbers* are the perfect ceilings of all ceiled sequences of rational numbers.

While the usual completeness axiom deals with *sets* of *real* numbers, this version deals with *sequences* of *rationals*. The modification dramatically improves the axiom’s usefulness in the calculus classroom. For one thing, students assimilate it easily into their ideas about approximation; I lead up to the completeness axiom with a discussion of how we might approximate the square root of 2 if we had no calculators, and why we believe there’s such a thing as the square root of 2 in the first place. More importantly, this version answers the question *Where do real numbers come from?* The usual version tells not what a real number is but only what you could expect to find if you were lucky enough to have some real numbers already on hand.

Students need plenty of examples to help them with completeness and the definition of a real number, but once those are digested, the class can immediately start to cut its logical teeth on the proofs of some simple propositions. For instance, *Every rational number is a real number also* (for any rational \( q \), consider the sequence \( q, q, q, \ldots \)). *For every positive real number, there is a rational number that is bigger* (either the given real number is itself rational, so that we can double it to yield a bigger rational number, or else the real number was the perfect ceiling of a sequence that already had some rational ceiling). And *every sequence of rational numbers that has a floor must have a perfect floor* (multiply the given sequence by \(-1\)). These can be posed to small groups as true-or-false items. Prompts can be given to help students answer the all-important question *Why?* After they mull things over, the instructor can step in to help the class chisel out respectable arguments.

Next to be proved is the usual version of completeness, with sequences instead of sets: *Every sequence of REAL numbers with a [REAL] ceiling has a PERFECT ceiling.* This is trickier to prove, but calculus students are receptive if the pictures are drawn nicely. Each real number in the given sequence is, by definition, the perfect ceiling of a sequence of *rational* numbers. We simply dovetail these sequences to create a single sequence of rationals. This sequence has a real ceiling and hence a rational ceiling (*for every positive real number, there is a rational number that is bigger*), so it has a perfect ceiling—which can be shown also to be the perfect ceiling for the original sequence of *real* numbers.

Two more needed results can be proved quickly in class or, even better, assigned as homework (with hints, of course):

**The capture theorem.** *If s is the perfect ceiling or perfect floor of a sequence, then any open interval containing s contains some element of the sequence.*

**The flipping-huge theorem.** *If a sequence of positive real numbers has 0 as its perfect floor, then there is no ceiling for the sequence of reciprocals of this sequence; and conversely.*

These results, while not surprising, are nice examples of reasoning in their own right. Their names help students retain them for use in the later proofs of weightier theorems, of which the IVT is only the first.
... and I am going to prove it to you

Time passes. The material described in the previous section is reviewed now and then, and finally continuity—the solid, precise, mathematical concept of continuity—is defined. As with the definition of real number, once students have digested continuity, they can be guided through arguments for some immediate propositions, including the following:

**The aura theorem.** Suppose that $f$ is continuous at $s$. If $f(s)$ is positive, then there is an open interval containing $s$ such that $f$ is positive on the entire interval. If $f(s)$ is negative, then there is an open interval containing $s$ such that $f$ is negative on the entire interval.

Then the class has enough preparatory material to assay a proof of the intermediate value theorem. The method used here, called proof by bisection, is not new—for instance the IVT is proved by bisection in Simpson’s reverse mathematics text [4, p. 87] and Mansfield’s real analysis text [2, pp. 109–111]—but it deserves to be better known, and it is appropriate for calculus students. It can be woven through the entire calculus course [5], used to prove the boundedness theorem and the extreme value theorem—two other theorems often dismissed as obvious because of a misunderstanding of the difference between continuity and continuity—and even used to prove the fundamental theorem itself.

The bisection proof is easy to present and allows for a dash of showmanship (imagine Bernard Bolzano channeled through Bob Barker—or through Drew Carey, for you younger calculus instructors). “Class,” says the instructor, “I’m thinking of a function $f$ that is continuous on the closed interval $[0, 1]$. I won’t show you its graph, but I will tell you that it is positive at $x = 0$ and negative at $x = 1$. Now, do you believe that the function $f$ ever takes the value 0 on this interval? ... You do? Then let’s play a game. You will guess an $x$ where you think $f(x)$ might be equal to 0. If you are right, you win. If you are wrong, I will tell you whether $f(x)$ is positive or negative, and you will guess again.

“The good news is that you may have as many guesses as you want. The bad news is that I don’t guarantee the function EVER takes the value 0, so you may have to guess forever. Have at it!”

Students hit upon the most efficient and natural approach almost immediately: guess $\frac{1}{2}$ first, then $\frac{1}{4}$ or $\frac{3}{4}$ depending on the sign of $f(\frac{1}{2})$, and so on. In this manner the class discovers an increasing sequence of points where $f$ is positive and a decreasing sequence of points where $f$ is negative. The first sequence must have a perfect ceiling, the second must have a perfect floor, these turn out to be the same real number, and that real number is exactly what we seek, that is, a $c$ such that $f(c) = 0$.

This proves only a special case of the IVT, a case usually referred to as “Bolzano’s Theorem” because he proved it in his 1817 paper; but it is easy to lead students to see that the proof would work for any interval $[a, b]$, any target value $p$, and either of the cases $f(a) < p < f(b)$ or $f(a) > p > f(b)$. Readers of this Journal will have no difficulty supplying the details of the proof; and more importantly, given the work leading up to that proof, their students will have little difficulty understanding those details. The key point is that nothing needs to be dismissed as obvious [6]. Even the claim that the perfect ceiling of the increasing sequence cannot be greater than the perfect floor of the decreasing one is worth a minute of proof rather than a second of hand-waving. Certainly the equality of those numbers deserves a proof, and that relies on a lemma:
To halve, and have naught. The perfect floor of the sequence \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \) is 0. In fact, if \( d \) is ANY positive real number, then the perfect floor of the sequence \( \frac{d}{2}, \frac{d}{4}, \frac{d}{8}, \ldots \) is 0.

The second part of the lemma follows from the first part, which in turn follows from the flipping-huge theorem and the fact that the sequence 2, 4, 8, \ldots has no ceiling. (If some student demands to see a proof that the sequence 2, 4, 8, \ldots has no ceiling, immediately sign her up as a mathematics major.)

Looking ahead

If we make proving the IVT a priority, the long-term benefits outweigh the short-term pains. Students learn a simple approach that drives the proofs of some fundamental theorems of analysis and makes some important concepts in calculus even more natural. This proof method becomes a common thread that they recognize and become comfortable with as the semester unfolds—reinforcing the definition of real number and resonating throughout the course, from the most fundamental notion in calculus to the fundamental theorem of calculus. The result is a new set (or rather a new sequence?) of opportunities for our students to exercise their reasoning abilities.

Do we accept these opportunities, or do we pass them up? After all, we can’t do everything. There always will be plenty of reasons to skip the proof of the IVT. Nonetheless, if we do present it without proof, we should at least observe a few guidelines:

- **DO** let students in on what a real number is.
- **DON’T** intimate that the IVT is obvious. This encourages—in fact demands—a misreading of the theorem.
- **DON’T** let students substitute their limited intuition for proof.
- **DON’T** suggest that the proof of the IVT is too difficult for mere mortals to understand.

With these points in mind, we can help our students cross that river after all—without being, mathematically, all wet.

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**Summary.** The Intermediate Value Theorem is not often proved in Calculus I classes because many teachers and students see the theorem as obvious and its proof as impenetrable. This article addresses those two misconceptions, showing how the IVT can be proved in Calc I … and why it should be.

**References**