TYCHONOFF'S THEOREM AND FILTERS

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Contents

Introduction

These notes record – and then moderately supplement – the final two lectures given in my Spring 2015 general topology course. The goal is to prove the following result.

Theorem 1. *(Tychonoff) For each $i \in I$, let X_i be a nonempty topological space, and let $X = \prod_{i \in I} X_i$, endowed with the product topology.*
a) The following are equivalent:
(i) Each X_i is quasi-compact.
(ii) X is quasi-compact.
b) The following are equivalent:
(i) Each X_i is compact.
(ii) X is compact.

Every implication except (i) $\implies$ (ii) in part a) is straightforward: let $i \in I$. Then $X_i = \pi_i(X)$, so if X is quasi-compact, so is X_i. Moreover, by the Slice Lemma, X_i is homeomorphic to a subspace of X, so if X is Hausdorff, so is X_i. Finally the product of Hausdorff spaces is Hausdorff. Henceforth by "Tychonoff's Theorem" we will mean the statement that a product of quasi-compact spaces is quasi-compact.

We will prove Tychonoff's theorem by developing a portion of the theory of convergence in topological spaces using filters. I worry that if I start by telling you what a filter is, it will be unnecessarily discouraging: the definition may not be immediately perspicuous. So instead I will begin with a **behavioral approach**: rather than telling you what filters are, I will first tell you what they do. In particular they swiftly reduce Tychonoff's Theorem to the conjunection of four facts.

Moreover, the behavior of filters has the ring of the familiar: many aspects are directly analogous to the behavior of sequences, either on the class of all topological spaces or on the class of first countable spaces. But in one place the analogy breaks down: there is a special kind of filter called an "ultrafilter", which plays a roughly similar role in the theory as passage to subsequences does in the theory of sequetnial convergence. However, the definition of an ultrafilter involves only the underlying set and not the topology at all, so it is as though in the sequential theory of all our sequences come "pre-extracted so as to converge". Of course this makes no sense in the sequential theory: at the level of sets, a subsequence of an arbitrary sequence is precisely an arbitrary sequence. However, it does make sense for filters, and in this point much of their magic resides.

1. What Should Filters Do?

1.1. The Behavior of Filters on Sets.

Let X be a set. A **filter** $\mathcal{F}$ on X is a subset of the power set 2^X, i.e., a set of subsets of X. In other words, it is the same sort of *structure* as a topology on X, but the properties that it is required to satisfy are different. I'm not teling you what they are yet!

As we have **ultrafilters**, which is an ultrafilter satisfying one additional property (to be named later!). You should try to think of a filter as acting like a sequence and of an ultrafilter as acting like a subsequence "but better".

In line with this analogy, if $x_\bullet : \mathbb{Z}^+ \to X$, $n \mapsto x_n$ is a sequence on X and $f : X \to Y$ is function, then we get a sequence on Y, $f(x_\bullet) : n \mapsto f(x_n)$. Similarly, if $\mathcal{F}$ is a filter on X and $f : X \to Y$ is a function, we can "push forward $\mathcal{F}$" to get a filter $f(\mathcal{F})$ on Y. The following property is a bit less familiar.

Fact 1. *Let $f : X \to Y$ be a map of sets, and let $\mathcal{F}$ be an ultrafilter on X. Then $f(\mathcal{F})$ is an ultrafilter on Y.*

Needless to say, we cannot prove Fact 1 yet: I haven't told you what an ultrafilter is! Similarly for the facts to come: we will circle back and prove them later.

1.2. The Behavior of Filters on Topological Spaces.

Now let X be a topological space, let $\mathcal{F}$ be a filter on X and let p be a point of X. Later we will define:

$\mathcal{F} \to x$. We read it as "$\mathcal{F}$ converges to p".

and

$\mathcal{F} \dashrightarrow p$. We read it as "$p$ is a limit point of $\mathcal{F}$."

We say that $\mathcal{F}$ is **convergent** if there exists $p \in X$ such that $\mathcal{F} \to p$.

Now come the last three facts:

Fact 2. *For a topological space X, the following are equivalent:*
(i) X is quasi-compact.
(ii) Every ultrafilter on X is convergent.

Fact 3. *Let X and Y be topological spaces, and let $f : X \to Y$ be a function. The following are equivalent:*
(i) f is continuous.
(ii) For all filters $\mathcal{F}$ on X and all $p \in X$, if $\mathcal{F} \to p$ then $f(\mathcal{F}) \to f(p)$.

Fact 4. *For $i \in I$, let X_i be a nonempty topological space, and let $X = \prod_{i \in I} X_i$, endowed with the product topology. For $i \in I$, let $\pi_i : X \to X_i$ be the projection map. Let $\mathcal{F}$ be a filter on X, and let p be a point of X. The following are equivalent:*
(i) We have $\mathcal{F} \to p$.
(ii) For all $i \in I$, we have $\pi_i(\mathcal{F}) \to \pi_i(p)$.

Fact 2 is somewhat reminiscent of the equivalence of compactness and sequential compactness in metric spaces, but it does not line up directly. The latter facts are much more recognizable: we saw that Fact 3 holds for sequences in any metric space, and more generally in any first countable space (but it does not hold in all topological spaces). Fact 4 does hold verbatim for sequences in any topological space; it just happens not to be as useful in an arbitrary topological space since the topology need not be determined by the convergent sequences (recall that there are spaces which are sequentially discrete but not discrete). Thus Facts 3 and 4 should be regarded as strong evidence that – whatever they may be! – filters work better than sequences for exploring the topology of an arbitrary topological space.

1.3. Tychonoff Proves Itself.

Let $\{X_i\}_{i \in I}$ be a family of nonempty quasi-compact topological spaces, and let $X = \prod_{i \in I} X_i$, endowed with the product topology. Let $\mathcal{F}$ be an ultrafilter on X. By Fact 1, for all $i \in I$, $\pi_i(\mathcal{F})$ is an ultrafilter on X_i. Since X_i is quasi-compact, by Fact 2 $\pi_i(\mathcal{F})$ converges, say to $p_i \in X_i$. Let $p \in X$ be the unique point such that $p_i = \pi_i(p)$ for all $i \in I$. By Fact 4 we have $\mathcal{F} \to p$. Thus every ultrafilter on X is convergent: applying Fact 2 again, we find that X is quasi-compact!

Note: If you lock yourself in a room with the four facts, you cannot escape this proof of Tychonoff's Theorem. Indeed all steps of the above proof were suggested by the students during the lecture.

2. Definitions

We are now suitably invested to see the definitions. Let X be a set.

A **filter** on X is a nonempty family $\mathcal{F}$ of nonempty subsets of X satisfying
(F1) For all $A_1, A_2 \in \mathcal{F}$ we have $A_1 \cap A_2 \in \mathcal{F}$.
(F2) If $A \in \mathcal{F}$ and $A \subset B \subset X$, then $B \in \mathcal{F}$.

An **ultrafilter** is a *maximal* filter on X, i.e., a filter $\mathcal{F}$ on X such that if $\mathcal{G}$ is a filter on X with $\mathcal{F} \subset \mathcal{G}$, then $\mathcal{F} = \mathcal{G}$.

Let X be a topological space, and let p be a point of X. The **neighborhood filter** at p, denoted $\mathcal{N}(p)$, is the set of all neighborhoods of p. Indeed it is a filter: if U_1, U_2 are both neighborhoods of p, then $p \in U_1^\circ$ and $p \in U_2^\circ$, so

$$p \in U_1^\circ \cap U_2^\circ \subset (U_1 \cap U_2)^\circ,$$

so $U_1 \cap U_2$ is a neighborhood of p. Futher, if $p \in U^\circ$ and $U \subset V \subset X$ then

$$p \in U^\circ \subset V^\circ,$$

so V is a neighborhood of p.

A filter $\mathcal{F}$ **converges to** p if $\mathcal{F} \supset \mathcal{N}(p)$, i.e., if every neighborhood of p is an element of $\mathcal{F}$. We denote this by $\mathcal{F} \rightarrow p$.

A point p **is a limit point** of a filter $\mathcal{F}$ if for every neighborhood U of p and every $A \in \mathcal{F}$ we have $U \cap A \neq \varnothing$. We denote this by $\mathcal{F} \dashrightarrow p$.

(Indeed $\mathcal{F} \rightarrow p$ implies $\mathcal{F} \dashrightarrow p$: if $\mathcal{F} \rightarrow p$, let U be a neighborhood of p. Then $U \in \mathcal{F}$, so for all $A \in \mathcal{F}$, we have $U \cap A \in \mathcal{F}$. But – crucially! – $\varnothing \notin \mathcal{F}$, so this means $U \cap A \neq \varnothing$.)

3. Proofs

3.1. The Zeroth Musketeer.

Well, we lied. We need some additional "foundational" information about filters. First, we need that every filter is contained in a maximal filter. The proof of this is easy if you have prior familiarity with Zorn's Lemma. If you have not seen Zorn's Lemma before, then you should look it up at some point, but it need not be now: you should feel free to regard part a) of the following fact as simply being a set-theoretic axiom.[1]

Fact 0. *Let $\mathcal{F}$ be a filter on a set X.*
a) There is an ultrafilter $\tilde{\mathcal{F}}$ on X with $\tilde{\mathcal{F}} \supset \mathcal{F}$.
b) Let $\mathcal{F}$ be a filter on X. If for a subset $Y \subset X$ we have $Y \cap A \neq \varnothing$ for all $A \in \mathcal{F}$, then there is a filter $\mathcal{G}$ on X containing $\mathcal{F} \cup \{A\}$.
c) For a filter on X, the following are equivalent:
(i) For every subset $Y \subset X$, we have $Y \in \mathcal{F}$ or $X \setminus Y \in \mathcal{F}$.
(N.B.: And not both, for then $\varnothing = Y \cap (X \setminus Y) \in \mathcal{F}$.)
(ii) $\mathcal{F}$ is an ultrafilter.

Proof. a) Let $\{\mathcal{F}_i\}_{i \in I}$ be a chain – i.e., totally ordered set – of filters on X. Then $\mathcal{F} = \bigcup_{i \in I}$ is a filter on X. By Zorn's Lemma, the partially ordered set of filters on X containing $\mathcal{F}$ has a maximal element.
b) One checks that under the given condition, the family $\mathcal{G} = \{B \supset Y \cap A \mid A \in \mathcal{F}\}$

[1]Indeed, from the perspective of a seasoned set theorist, it is *better* to regard it this way, since in fact the existence of an ultrafilter containing a filter is known to be *a weaker axiom* than the one which is used to prove it: Zorn's Lemma is equivalent to the Axiom of Choice.

is a filter on X. Let $A \in \mathcal{F}$; then $Y \cap A \in \mathcal{G}$ so also $Y \supset Y \cap A \in \mathcal{G}$; similarly, for all $A \in \mathcal{F}$, we have $A \supset Y \cap A \in \mathcal{G}$, so $\mathcal{G}$ contains $\mathcal{F} \cup \{A\}$.
c) (i) $\implies$ (ii): If $\mathcal{F}$ has the property that for any subset $Y \subset X$ either $Y \in \mathcal{F}$ or $X \setminus Y \in \mathcal{F}$, then any larger filter would have to contain *both* Y and $X \setminus Y$ for some $Y \subset X$. This cannot happen: then $\varnothing = Y \cap (X \setminus Y) \in \mathcal{F}$.
(ii) $\implies$ (i): Let $Y \subset X$ be a subset. If there were $A_1, A_2 \in \mathcal{F}$ such that $Y \cap A_1 = (X \setminus Y) \cap A_2 = \varnothing$, then $A_1 \cap A_2 = \varnothing$, contradiction. So we must have either that $Y \cap A \neq \varnothing$ for all $A \in \mathcal{F}$ or $(X \setminus Y) \cap A \neq \varnothing$ for all $A \in \mathcal{F}$. Applying part b), we get that there is a filter $\mathcal{G}$ containing $\mathcal{F}$ such that either $Y \in \mathcal{G}$ or $X \setminus Y \in \mathcal{G}$. But $\mathcal{F}$ is an ultrafilter, so $\mathcal{F} = \mathcal{G}$. □

3.2. **Proof of Fact 1.**

Let $\mathcal{F}$ be an ultrafilter on X, let $f : X \to Y$ be a function, and let

$$f(\mathcal{F}) = \{B \supset f(A) \mid A \in \mathcal{F}\}.$$

We will use the characterization of ultrafilters given by Fact 0: let $W \subset Y$. Then

$$f^{-1}(Y \setminus W) = X \setminus f^{-1}(W).$$

Since $\mathcal{F}$ is an ultrafilter, we thus have either $f^{-1}(W) \in \mathcal{F}$ or $f^{-1}(Y \setminus W) \in \mathcal{F}$. In the former case we have

$$W \supset f(f^{-1}(W)) \in f(\mathcal{F}),$$

whereas in the latter case we have

$$Y \setminus W \supset f(f^{-1}(Y \setminus W)) \in f(\mathcal{F}).$$

So $f(\mathcal{F})$ is an ultrafilter on Y.

3.3. **Proof of Fact 2.**

A family $\{A_i\}_{i\in I}$ of subsets of a set X satisfies the **finite intersection property** if for all finite subsets $J \subset I$ we have $\bigcap_{i\in J} A_i \neq \varnothing$.

Quasi-compactness can be expressed in terms of closed subsets as follows: for any family $\{A_i\}_{i\in I}$ of closed subsets of X satisfying the finite intersection condition we have $\bigcap_{i\in I} A_i \neq \varnothing$.

(i) $\implies$ (ii): Suppose X is quasi-compact, and let $\mathcal{F}$ be an ultrafilter on X. The elements of $\mathcal{F}$ satisfy the finite intersection property, hence so too does $\{\overline{A} \mid A \in \mathcal{F}\}$. By quasi-compactness, there is $p \in \bigcap_{A\in\mathcal{F}} \overline{A}$. We claim $\mathcal{F} \to p$. Indeed, since $\mathcal{F}$ is an ultrafilter, by Fact 0b) above, it is enough to show that for any neighborhood U of p and all $A \in \mathcal{F}$ we have $U \cap A \neq \varnothing$. Since $p \in \overline{A}$, every neighborhood of p meets A. Okay, so U does: $U \cap A \neq \varnothing$.

(ii) $\implies$ (i): Suppose that every ultrafilter on X is convergent, and let $\mathcal{A} = \{A_i\}_{i\in I}$ be a family of closed subsets satisfying the finite intersection property. Then $\mathcal{F} = \{B \supset \bigcap_{i\in J} A_i \mid J \subset I \text{ is finite }\}$ is a filter on X containing $\mathcal{A}$; by Fact 0 there is an ultrafilter $\tilde{\mathcal{F}}$ on X containing $\mathcal{A}$. By asumption, there is a point $p \in X$ such that $\tilde{\mathcal{F}} \to p$. Fix $i \in I$, and let U be a neighborhood of p. Then since $U, A_i \in \tilde{\mathcal{F}}$ we have $U \cap A_i \neq \varnothing$. Thus A_i meets every neighborhood of p, so $p \in \overline{A_i} = A_i$. It follows that $p \in \bigcap_{i\in I} A_i$. So X is quasi-compact.

3.4. **Proof of Fact 3.**

This is an analogue for filters of a very basic and familiar fact for sequences in metric spaces. It thus makes a very good exercise: I leave it you!

3.5. **Proof of Fact 4.**

Similarly, this is an analogue of a very familiar fact for sequences (in any topological space, this time). Again: a good exercise!

4. Debriefing

4.1. **Can I get an ultrafilter?**

Now that ultrafilters have proven a big theorem for us, we might like to meet one in person. It turns out though that the only examples of ultrafilters that we can get our hands on are the uninteresting ones.

Let X be a set, and let $\varnothing \neq Y \subset X$ be a nonempty subset. Then the family $\mathcal{F}_Y = \{Y \subset B \subset X\}$ of all subsets containing Y is a filter on X. (In particular $\mathcal{F}_X = \{X\}$ is the unique minimal filter on X.) For subsets $Y_1, Y_2 \subset X$ we have

$$Y_1 \subset Y_2 \iff \mathcal{F}_{Y_1} \supset \mathcal{F}_{Y_2}.$$

Since Y needs to be nonempty, the maximal elements among such filters occur when $Y = \{p\}$ is a single point: then $\mathcal{F}_p$ is the set of all subsets $A \subset X$ with $p \in X$. Notice that for any subset $Y \subset X$, exactly one of Y and $X \setminus Y$ contains p, so $\mathcal{F}_p$ is not just maximal among the family of filters we were considering but is actually an ultrafilter. Such ultrafilters are called **principal**.

If X is finite, then $Y = \bigcap_{A \in \mathcal{F}} A$ is a finite intersection of elements of $\mathcal{F}$ so must be an element of $\mathcal{F}$. A moment's thought shows $\mathcal{F} = \mathcal{F}_Y$. In particular all ultrafilters on a finite set are principal. (So there is nothing interesting happening here. Luckily, the quasi-compactness of a finite topological space is similarly uninteresting!)

Now suppose X is infinite. Then the family $\mathcal{F}_F$ of all cofinite subsets $Y \subset X$ – i.e., with $X \setminus Y$ finite – is a filter on X, called the **Frechet filter**. Since X admits subsets Y such that neither Y nor $X \setminus Y$ is finite, by Fact 0 the Frechet filter is *not* an ultrafilter. We claim that an ultrafilter $\mathcal{F}$ on X is nonprincipal – i.e., not of the form $\mathcal{F}_p$ for some $p \in X$ – iff it contains the Frechet filter. On the one hand, for all $p \in X$, $X \setminus \{p\} \in \mathcal{F}_F$ so we cannot have $\{p\} \in \mathcal{F}_F$, hence not in any ultrafilter $\mathcal{F} \supset \mathcal{F}_F$. Conversely, to say that an ultrafilter $\mathcal{F}$ is nonprincipal is to say that for all $p \in X$ $\{p\}$ is not in $\mathcal{F}$, so by Fact 0 $X \setminus \{p\}$ must be. The family of sets obtained by taking finite intersections of $\{X \setminus \{p\} \mid p \in X\}$ is precisely $\mathcal{F}_F$.

It now follows from Fact 0 that nonprincipal ultrafilters exist: indeed, we can extend $\mathcal{F}_F$ to an ultrafilter, and any such ultrafilter is nonprincipal. In fact it can be shown that the number of nonprincipal ultrafilters on any infinite set X is $2^{2^{\# X}}$ – the cardinality of the family of all families of subsets of X. Well, take $X = \mathbb{Z}^+$: if there are $2^{2^{\aleph_0}} = 2^{\mathfrak{c}}$ nonprincipal ultrafilters on X, you should be able to show me one right? Good luck with that: let me know if you succeed. The point is that Zorn's Lemma is allowing us to assert the existence of something which is in practice impossible to get our hands on.

4.2. **Filter bases, filter subbases and limit points.**

In the above presentation, we hewed close to saying the minimum possible amount about the theory of convergence via filters needed to establish Tychonoff's Theorem. There are virtues in that – especially, the proof fit in two lectures! – but it is also true that some efficiency was gained simply by not properly explaining what we're doing. If we go back over a few points in more detail we can see more of the general theory. In particular, notice that we introduced the notion of a limit point of a filter then didn't use it...explicitly. But really we did, as we will now see.

First, a nonempty collection F of nonempty subsets of a set X is a **filter base** if
(FB) For all $A_1, A_2 \in F$, there is $A_3 \in F$ with $A_3 \subset A_1 \cap A_2$.

This condition on a family of sets is necessary and sufficient for

$$\mathcal{F}(F) = \{A \subset B \subset X \mid A \in F\}$$

to be a filter on X, as is easy to check. In fact, for a filter base F we have that $\mathcal{F}(F)$ is the intersection of all filters containing F.

As one example of this concept, we revisit the "push forward". Let $f : X \to Y$ be a map, and let F be a filter base on X. Then

$$f(F) = \{f(A) \mid A \in F\}$$

is a filter base on Y: for $B_1, B_2 \in f(F)$, write $B_1 = f(A_1)$ and $B_2 = f(A_2)$: then there is $A_3 \in F$ with $A_3 \subset A_1 \cap A_2$ and then

$$f(A_3) \subset f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2).$$

Thus the concept of pushing forward is a little easier on filter bases, but in general the push forward of a filter is only a filter base: for instance let

$$f : \{0\} \to \mathbb{R}$$

be the inclusion map. Then the (unique!) (ultra)filter $\mathcal{F} = \{\{0\}\}$ on $\{0\}$ "naively pushes forward" to $\{\{0\}\}$ on $\mathbb{R}$, and this is not a filter, but the filter it generates is the principal ultrafilter $\mathcal{F}_0$. (On the other hand, it is easy to see that if $f : X \to Y$ is surjective, then the "naive pushforward" of a filter is already a filter, and the "naive pushforward" of an ultrafilter is an ultrafilter. In fact to prove Tychonoff's Theorem we need to push forward via the projection maps $\pi_i : \prod_{i \in I} X_i \to X_i$, which are surjective. So we could have used the "naive pushforward" here...but that seems silly.)

We can take things a step further. A nonempty collection of nonempty subsets $\mathcal{A} = \{A_i\}_{i \in I}$ of X is a **filter subbase** if it satisfies the finite intersection property. Clearly this condition is necessary for $\mathcal{A}$ to be contained in any filter. It is also sufficient: assuming that finite intersections are nonempty, then

$$F(\mathcal{A}) = \{\bigcap_{i \in J} A_i \mid J \subset I \text{ is finite}\}$$

is a family of nonempty sets which is closed under finite intersections, and thus is certainly a filter base, and thus $\mathcal{F}(F(\mathcal{A}))$ is a filter containing it – again, the smallest filter that contains it.

Notice that the situation here is more interesting than for topologies: *any* family of subsets of a set X generates a topology by taking finite intersections (regarding X itself as the "empty intersection") then arbitrary unions. We say that two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of a set X are **compatible** if there is a filter $\mathcal{F}$ containing both of them: equivalently, $\mathcal{A} \cup \mathcal{B}$ satisfies the finite intersection condition. If $\mathcal{A}$ and $\mathcal{B}$ are both filters, then the finite intersection condition is equivalent to: for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$ we have $A \cap B \neq \varnothing$.

Now observe: if X is a topological space and $\mathcal{F}$ is a filter on X, then a point p of X is a limit point of $\mathcal{F}$ precisely when the neighborhood filter $\mathcal{N}(p)$ is compatible with $\mathcal{F}$. It is now easy to see:

Fact 5. *Let X be a topological space, and let p be a point of X.*
a) For a filter $\mathcal{F}$ on X, $\mathcal{F} \dashrightarrow p$ iff there is an ultrafilter $\tilde{\mathcal{F}} \supset \mathcal{F}$ such that $\tilde{\mathcal{F}} \to p$. It follows that:
b) For an ultrafilter $\mathcal{F}$, we have $\mathcal{F} \to p$ iff $\mathcal{F} \dashrightarrow p$.
c) A topological space is quasi-compact iff every filter on X has a limit point.

If you look back at the proof of Fact 2, you'll see that we essentially established the criterion that quasi-compactness means that every filter has a limit point without explicitly saying so. This latter criterion also clarifies the sense in which filters "repair" sequences: although sequential compactness coincides with quasi-compactness in metric spaces, in general there are (even Hausdorff) topological spaces which are sequentially compact but not quasi-compact *and* there are (even Hausdorff) topological spaces which are quasi-compact but not sequentially compact. But if we use filters instead of sequences, all is well.

4.3. **This is Really the End.**

The theory of filters was developed by Henri Cartan and exposed by him in two notes in the French Academy of Sciences [Ca37]. It was swiftly incorporated into the topology text of the collective mathematical entity Nicolas Bourbaki [Bo]. Bourbaki's text develops the theory of filters in amazing detail: much of the subject of general topology gets factored through it.

You should be aware that there is a rival theory of convergence in topological spaces: **nets**. Nets were developed by E.H. Moore in the early 20th century [Mo10], [Mo15] and further developed in a paper with his student H. L. Smith [MS22]. The emphasis in these papers was on a more general limiting process – so that e.g. the notoriously rather complicated sense in which Riemann sums converge to the Riemann integral becomes a special case of net convergence – rather than general topological spaces: much of the work of Moore and Smith takes place in normed linear spaces. A theory of convergence in general topological spaces was developed by Birkhoff [Bi37], Tukey [T] and Kelley [Ke50].

Nets bear a closer resemblance to sequences than filters: a net in a set X is a map from a certain kind of partially ordered set into X, so the order relation is still present. In some areas of mathematics – e.g. functional analysis – this closeness becomes a (moderate) advantage. For instance, if one is trying to adapt arguments from a Banach space (which is a metric space) to the case of a topological vector space (which need not be first countable and is not in many cases of intereset), you have a sequential argument in mind that you'd like to change as little as possible, and nets make this changeover process especially routine.

Distressingly, the historical fact that filters were created by Europeans and nets by Americans is to a large extent preserved in the standard texts: it is relatively rare to find English language topology texts that treat filters and rarer still to find texts which treat both and compare them: a notable exception is [Wi]. (Willard's text is not always the most gripping read, but it is probably more inclusive and reliable in its coverage of material than any other text currently in print.)

The theory of nets, filters and relations between them lies at the center of my own topological expertise. Several years ago I wrote a substantial manuscript on this material: [C]. It is written at a moderately higher level, and more things have been left as exercises for the reader than in the treatment given here. But if you are still reading this, you must be very interested indeed, and perhaps you would enjoy the more comprehensive treatment.

The other advantage of filters over nets is that nets are *only* useful for the study of convergence in topological spaces, whereas filters are useful – increasingly so! – in mathematics as a whole. The entire edifice of *nonstandard analysis* for instance, is built on "transfer principles" using ultrafilters. For a brief introduction to the use of ultrafilters in mathematical logic – including their use to prove the basic **compactness theorem** (!! – yes, *it is* related to Tychonoff's Theorem) in this area, see for instance [M, §6].

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