

§5: (NEIGHBORHOOD) SUB/BASES

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We have found our way to an important definition: if τ is a topology on X and $\mathcal{F} \in 2^{2^X}$ is a family such that $\tau = \tau(\mathcal{F})$, we say \mathcal{F} is a **subbase** (or **subbasis**) for τ .

Example: Let X be a set of cardinality at least 2.

(a) Again, if we take \mathcal{F} to be the empty family, then $\tau(\mathcal{F})$ is the indiscrete topology.
 (b) If Y is a subset of X and we take $\mathcal{F} = \{Y\}$, then the open sets in the induced topology τ_Y are precisely those which contain Y . Note that these 2^X topologies are all distinct. If $Y = X$ this again gives the indiscrete topology, whereas if $Y = \emptyset$ we get the discrete topology. Otherwise we get a non-Hausdorff topology: indeed for $x \in X$, $\{x\}$ is closed iff $x \in X \setminus Y$.

Exercise: Let X be a set and Y, Y' be two subsets of X . Show that TFAE:

- (i) (X, τ_Y) is homeomorphic to $(X, \tau_{Y'})$.
- (ii) $\#Y = \#Y'$.

The nomenclature “subbase” suggests the existence of a cognate concept, that of a “base”. Based upon our above intrinsic construction of $\tau(\mathcal{F})$, it would be reasonable to guess that \mathcal{F}_1 is a base, or more precisely that a basis for a topology should be a collection of open sets, closed under finite intersection, whose unions recover all the open sets. But it turns out that a weaker concept is much more useful.

Consider the following axioms on a family \mathcal{B} of subsets of a set X :

- (B1) $\forall U_1, U_2 \in \mathcal{B}$ and $x \in U_1 \cap U_2$, $\exists U_3 \in \mathcal{B}$ such that $x \in U_3 \subset U_1 \cap U_2$.
- (B2) For all $x \in X$, there exists $U \in \mathcal{B}$ such that $x \in U$.

The point here is that (B1) is weaker than the property of being closed under finite intersections, but is just as good for constructing the generated topology:

Proposition 1. *Let $\mathcal{B} = (\mathcal{U}_i)_{i \in I}$ be a family of subsets of X satisfying (B1) and (B2). Then $\tau(\mathcal{B})$, the topology generated by \mathcal{B} , is given by $\{\bigcup_{i \in J} \mathcal{U}_i \mid J \subset I\}$, or in other words by the collection of arbitrary unions of elements of \mathcal{B} .*

Proof: Let T be the set of arbitrary unions of elements of \mathcal{B} ; certainly $T \subset \tau(\mathcal{B})$. It is automatic that $\emptyset \in T$ (take the empty union), and (B2) guarantees that $X = \bigcup_{i \in I} \mathcal{U}_i$. Clearly T is closed under arbitrary unions, so it suffices to show that the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ of any two elements of \mathcal{B} can be expressed as a union over some set of elements of \mathcal{B} . But the point is that (B1) visibly guarantees this: for each $x \in \mathcal{U}_1 \cap \mathcal{U}_2$, by (B1) we may choose $\mathcal{U}_x \in \mathcal{B}$ such that $x \in \mathcal{U}_x \subset \mathcal{U}_1 \cap \mathcal{U}_2$. Then

$$\mathcal{U}_1 \cap \mathcal{U}_2 = \bigcup_{x \in \mathcal{U}_1 \cap \mathcal{U}_2} \mathcal{U}_x.$$

A family \mathcal{B} of subsets of X satisfying (B1) and (B2) is said to be a **base** (or **basis**) for the topology it generates. Or, to put it another way, a subcollection \mathcal{B} of the open sets of a topological space (X, τ) which satisfies (B1) and (B2) is called a base, and then every open set is obtained as a union of elements of the base. And conversely:

Exercise X.X: Let (X, τ) be a topological space and \mathcal{B} be a family of open sets. Suppose that every open set in X may be written as a union of elements of \mathcal{B} . Show that \mathcal{B} satisfies (B1) and (B2).

Example X.X: In a metric space (X, d) , then open balls form a base for the topology: especially, the intersection of two open balls need not be an open ball but contains an open ball about each of its points. Indeed, the open balls with radii $\frac{1}{n}$, for $n \in \mathbb{Z}^+$, form a base.

Example X.X: In \mathbb{R}^d , the d -fold products $\prod_{i=1}^d (a_i, b_i)$ of open intervals with rational endpoints is a base. In particular this shows that \mathbb{R}^d has a **countable base**, which will turn out to be an extremely nice property for a topological space to have.

Exercise X.X: a) On \mathbb{R} , show that intervals of the form $[a, b)$ form a base for a topology τ_S which is strictly finer than the standard (metric) topology on \mathbb{R} . The space (\mathbb{R}, τ_S) is called the **Sorgenfrey line** after Robert Sorgenfrey.¹

b) Show that the Sorgenfrey line does *not* have a countable base.

0.1. **Neighborhood bases.** Let x be a point of a topological space X . A family $\{N_\alpha\}$ of neighborhoods of x is said to be a **neighborhood base at x** (or a **fundamental system of neighborhoods of x**) if every neighborhood N of x contains some N_α . Suppose we are given for each $x \in X$ a neighborhood base \mathcal{B}_x at x . The following axioms hold:

(NB1) $N \in \mathcal{B}_x \implies x \in N$.

(NB2) $N, N' \in \mathcal{B}_x \implies$ there exists N'' in \mathcal{B}_x such that $N'' \subset N \cap N'$.

(NB3) $N \in \mathcal{B}_x \implies$ there exists $V \in \mathcal{B}_x, V \subset N$, such that $y \in V \implies V \in \mathcal{B}_y$.

Conversely:

Proposition 2. *Suppose given a set X and, for each $x \in X$, a collection \mathcal{B}_x of subsets satisfying (NB1)-(NB3). Then the collections $\mathcal{N}_x = \{Y \mid \exists N \in \mathcal{B}_x \mid Y \supset N\}$ are the neighborhood systems for a unique topology on X , in which a subset U is open iff $x \in U \implies U \in \mathcal{N}_x$. Each \mathcal{N}_x is a neighborhood basis at x .*

Exercise X.X: Prove Proposition 2.

Remark: Consider the condition

(NB3') $N \in \mathcal{B}_x, y \in N \implies N \in \mathcal{B}_y$.

¹The merit of this “weird” topology is that it is often a source of counterexamples.

Replacing (NB3) with (NB3') amounts to restricting attention to open neighborhoods. Since (NB3') \implies (NB3), we may specify a topology on X by giving, for each x , a family \mathcal{N}_x of sets satisfying (NB1), (NB2), (NB3'). This is a very convenient way to define a topology: e.g. the metric topology is thus defined just by taking \mathcal{N}_x to be the family $\{B(x, \epsilon)\}$ of ϵ balls about x .

Here is a more interesting example. Let $M = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. Now:

For $P = (x, y) \in M$ with $y > 0$, we take \mathcal{B}_P to be the set of Euclidean-open disks $B(P, r)$ centered at P with radius $r \leq y$ (so that $B(P, r) \subset M$).

For $P = (x, 0) \in M$, we take \mathcal{B}_P to be the family of sets $\{P \cup D((x, y), y) \mid y > 0\}$; in other words, an element of \mathcal{B}_P consists of an open disk in the upper half plane which is tangent to the x -axis at P , together with P .

Exercise: Verify that $\{\mathcal{B}_P \mid P \in M\}$ satisfies (NB1), (NB2) and (NB3'), so there is a unique topology τ_M on M with these sets as neighborhood bases. The space (M, τ_M) is called the **Moore-Niemytzki plane**.²

Proposition 3. *Suppose that $\varphi : X \rightarrow X$ is a self-homeomorphism of the topological space X . Let $x \in X$ and \mathcal{N}_x be a neighborhood basis at x . Then $\varphi(\mathcal{N}_x)$ is a neighborhood basis at $y = \varphi(x)$.*

Proof: It suffices to work throughout with open neighborhoods. Let V be an open neighborhood of y . By continuity, there exists an open neighborhood U of x such that $\varphi(U) \subset V$. Since φ^{-1} is continuous, $\varphi(U)$ is open.

As for any category, the automorphisms of a topological space X form a group, $\text{Aut}(X)$. We say X is **homogeneous** if $\text{Aut}(X)$ acts transitively on X , i.e., for any $x, y \in X$ there exists a self-homeomorphism φ such that $\varphi(x) = y$. By the previous proposition, if a space is homogeneous we can recover the entire topology from the neighborhood basis of a single point. In particular this applies to topological groups.

Finally, one can also define the concept of a **neighborhood subbase**. We have no particular need of this in the sequel, so we leave the precise definition to the interested reader.

²Like the Sorgenfrey line, and possibly even more so, this space is extremely useful for showing nonimplications among topological properties.