§5: (NEIGHBORHOOD) SUB/BASES

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We have found our way to an important definition: if \( \tau \) is a topology on \( X \) and \( \mathcal{F} \in 2^X \) is a family such that \( \tau = \tau(\mathcal{F}) \), we say \( \mathcal{F} \) is a \textbf{subbase} (or \textbf{subbasis}) for \( \tau \).

Example: Let \( X \) be a set of cardinality at least 2.
(a) Again, if we take \( \mathcal{F} \) to be the empty family, then \( \tau(\mathcal{F}) \) is the indiscrete topology.
(b) If \( Y \) is a subset of \( X \) and we take \( \mathcal{F} = \{ Y \} \), then the open sets in the induced topology \( \tau_Y \) are precisely those which contain \( Y \). Note that these \( 2^X \) topologies are all distinct. If \( Y = X \) this again gives the indiscrete topology, whereas if \( Y = \emptyset \) we get the discrete topology. Otherwise we get a non-Hausdorff topology: indeed for \( x \in X \), \( \{ x \} \) is closed iff \( x \in X \setminus Y \).

Exercise: Let \( X \) be a set and \( Y, Y' \) be two subsets of \( X \). Show that TFAE:
(i) \((X, \tau_Y)\) is homeomorphic to \((X, \tau_{Y'})\).
(ii) \( \#Y = \#Y' \).

The nomenclature “subbase” suggests the existence of a cognate concept, that of a “base”. Based upon our above intrinsic construction of \( \tau(\mathcal{F}) \), it would be reasonable to guess that \( \mathcal{F} \) is a base, or more precisely that a basis for a topology should be a collection of open sets, closed under finite intersection, whose unions recover all the open sets. But it turns out that a weaker concept is much more useful.

Consider the following axioms on a family \( \mathcal{B} \) of subsets of a set \( X \):

(B1) \( \forall U_1, U_2 \in \mathcal{B} \) and \( x \in U_1 \cap U_2 \), \( \exists U_3 \in \mathcal{B} \) such that \( x \in U_3 \subset U_1 \cap U_2 \).
(B2) For all \( x \in X \), there exists \( U \in \mathcal{B} \) such that \( x \in U \).

The point here is that (B1) is weaker than the property of being closed under finite intersections, but is just as good for constructing the generated topology:

**Proposition 1.** Let \( \mathcal{B} = (\mathcal{U}_i)_{i \in I} \) be a family of subsets of \( X \) satisfying (B1) and (B2). Then \( \tau(\mathcal{B}) \), the topology generated by \( \mathcal{B} \), is given by \( \{ \bigcup_{J \subset I} \mathcal{U}_i \} \), or in other words by the collection of arbitrary unions of elements of \( \mathcal{B} \).

Proof: Let \( T \) be the set of arbitrary unions of elements of \( \mathcal{B} \); certainly \( T \subset \tau(\mathcal{B}) \). It is automatic that \( \emptyset \in T \) (take the empty union), and (B2) guarantees that \( X = \bigcup_{i \in I} \mathcal{U}_i \). Clearly \( T \) is closed under arbitrary unions, so it suffices to show that the intersection \( \mathcal{U}_{i_1} \cap \mathcal{U}_{i_2} \) of any two elements of \( \mathcal{B} \) of \( \mathcal{B} \) can be expressed as a union over some set of elements of \( \mathcal{B} \). But the point is that (B1) visibly guarantees this: for each \( x \in \mathcal{U}_{i_1} \cap \mathcal{U}_{i_2} \), by (B1) we may choose \( \mathcal{U}_x \in \mathcal{B} \) such that \( x \in \mathcal{U}_x \subset \mathcal{U}_{i_1} \cap \mathcal{U}_{i_2} \). Then

\[
\mathcal{U}_{i_1} \cap \mathcal{U}_{i_2} = \bigcup_{x \in \mathcal{U}_{i_1} \cap \mathcal{U}_{i_2}} \mathcal{U}_x.
\]
A family \( \mathcal{B} \) of subsets of \( X \) satisfying (B1) and (B2) is said to be a base (or basis) for the topology it generates. Or, to put it another way, a subcollection \( \mathcal{B} \) of the open sets of a topological space \( (X, \tau) \) which satisfies (B1) and (B2) is called a base, and then every open set is obtained as a union of elements of the base. And conversely:

Exercise X.X: Let \( (X, \tau) \) be a topological space and \( \mathcal{B} \) be a family of open sets. Suppose that every open set in \( X \) may be written as a union of elements of \( \mathcal{B} \). Show that \( \mathcal{B} \) satisfies (B1) and (B2).

Example X.X: In a metric space \( (X, d) \), then open balls form a base for the topology: especially, the intersection of two open balls need not be an open ball but contains an open ball about each of its points. Indeed, the open balls with radii \( \frac{1}{n} \), for \( n \in \mathbb{Z}^+ \), form a base.

Example X.X: In \( \mathbb{R}^d \), the \( d \)-fold products \( \prod_{i=1}^d (a_i, b_i) \) of open intervals with rational endpoints is a base. In particular this shows that \( \mathbb{R}^d \) has a countable base, which will turn out to be an extremely nice property for a topological space to have.

Exercise X.X: a) On \( \mathbb{R} \), show that intervals of the form \( (a, b) \) form a base for a topology \( \tau_S \) which is strictly finer than the standard (metric) topology on \( \mathbb{R} \). The space \( (\mathbb{R}, \tau_S) \) is called the Sorgenfrey line after Robert Sorgenfrey.\(^1\) b) Show that the Sorgenfrey line does not have a countable base.

0.1. Neighborhood bases. Let \( x \) be a point of a topological space \( X \). A family \( \{N_\alpha\} \) of neighborhoods of \( x \) is said to be a neighborhood base at \( x \) (or a fundamental system of neighborhoods of \( x \)) if every neighborhood \( N \) of \( x \) contains some \( N_\alpha \). Suppose we are given for each \( x \in X \) a neighborhood base \( \mathcal{B}_x \) at \( x \). The following axioms hold:

\[(NB1) \ N \in \mathcal{B}_x \implies x \in N.\]
\[(NB2) \ N, N' \in \mathcal{B}_x \implies \text{there exists } N'' \in \mathcal{B}_x \text{ such that } N'' \subset N \cap N'.\]
\[(NB3) \ N \in \mathcal{B}_x \implies \text{there exists } V \in \mathcal{B}_x, V \subset N, \text{ such that } y \in V \implies V \in \mathcal{B}_y.\]

Conversely:

**Proposition 2.** Suppose given a set \( X \) and, for each \( x \in X \), a collection \( \mathcal{B}_x \) of subsets satisfying (NB1)-(NB3). Then the collections \( \mathcal{N}_x = \{ Y \mid \exists N \in \mathcal{B}_x \mid Y \supset N \} \) are the neighborhood systems for a unique topology on \( X \), in which a subset \( U \) is open iff \( x \in U \implies U \in \mathcal{N}_x \). Each \( \mathcal{N}_x \) is a neighborhood basis at \( x \).

Exercise X.X: Prove Proposition 2.

Remark: Consider the condition

\[(NB3') \ N \in \mathcal{B}_x, y \in N \implies N \in \mathcal{B}_y.\]

\(^1\)The merit of this “weird” topology is that it is often a source of counterexamples.
Replacing (NB3) with (NB3′) amounts to restricting attention to open neighborhoods. Since (NB3′) ⇒ (NB3), we may specify a topology on \( X \) by giving, for each \( x \), a family \( N_x \) of sets satisfying (NB1), (NB2), (NB3′). This is a very convenient way to define a topology: e.g. the metric topology is thus defined just by taking \( N_x \) to be the family \( \{B(x, \epsilon)\} \) of \( \epsilon \) balls about \( x \).

Here is a more interesting example. Let \( M = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} \). Now:

For \( P = (x, y) \in M \) with \( y > 0 \), we take \( B_P \) to be the set of Euclidean-open disks \( B(P, r) \) centered at \( P \) with radius \( r \leq y \) (so that \( B(P, r) \subset M \)).

For \( P = (x, 0) \in M \), we take \( B_P \) to be the family of sets \( \{P \cup D((x, y), y) \mid y > 0\} \); in other words, an element of \( B_P \) consists of an open disk in the upper half plane which is tangent to the \( x \)-axis at \( P \), together with \( P \).

Exercise: Verify that \( \{B_P \mid P \in M\} \) satisfies (NB1), (NB2) and (NB3′), so there is a unique topology \( \tau_M \) on \( M \) with these sets as neighborhood bases. The space \((M, \tau_M)\) is called the Moore-Niemytzki plane.\(^2\)

Proposition 3. Suppose that \( \varphi : X \to X \) is a self-homeomorphism of the topological space \( x \). Let \( x \in X \) and \( N_x \) be a neighborhood basis at \( x \). Then \( \varphi(N_x) \) is a neighborhood basis at \( y = \varphi(x) \).

Proof: It suffices to work throughout with open neighborhoods. Let \( V \) be an open neighborhood of \( y \). By continuity, there exists an open neighborhood \( U \) of \( x \) such that \( \varphi(U) \subset V \). Since \( \varphi^{-1} \) is continuous, \( \varphi(U) \) is open.

As for any category, the automorphisms of a topological space \( X \) form a group, \( \text{Aut}(X) \). We say \( X \) is homogeneous if \( \text{Aut}(X) \) acts transitively on \( X \), i.e., for any \( x, y \in X \) there exists a self-homeomorphism \( \varphi \) such that \( \varphi(x) = y \). By the previous proposition, if a space is homogeneous we can recover the entire topology from the neighborhood basis of a single point. In particular this applies to topological groups.

Finally, one can also define the concept of a neighborhood subbase. We have no particular need of this in the sequel, so we leave the precise definition to the interested reader.

\(^2\)Like the Sorgenfrey line, and possibly even more so, this space is extremely useful for showing nonimplications among topological properties.