

§4: THE LATTICE OF ALL TOPOLOGIES ON A GIVEN SET

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Let X be a set, and let $\text{Top}(X) \subset 2^{2^X}$ be the collection of all topologies on X .

Exercise X.X.X*: Suppose X is infinite. Show that $\#\text{Top}(X) = 2^{2^X}$.

As a subset of 2^{2^X} , $\text{Top}(X)$ inherits a partial ordering: we define $\tau_1 \leq \tau_2$ if $\tau_1 \subset \tau_2$, i.e., if every τ_1 -open set is also τ_2 -open.

If $\tau_1 \leq \tau_2$ we say that τ_1 is **coarser** than τ_2 and also that τ_2 is **finer** than τ_1 .¹ We say that two topologies are **comparable** if one of them is coarser than the other.

Exercise X.X: Let $\mathcal{T} \subset 2^{2^X}$ be any family of topologies on X . Then $\bigcap_{\tau \in \mathcal{T}} \tau$ is a topology on X . (By convention, $\bigcap_{\emptyset} = 2^X$ is the discrete topology.)

Let $\mathcal{F} \in 2^{2^X}$ be any family of subsets of X . Then among all topologies τ on X containing \mathcal{F} there is a coarsest topology $\tau(\mathcal{F})$, namely the intersection of all topologies containing \mathcal{F} . (**Tournant dangereuse**: here $\tau(\emptyset) = \{\emptyset, X\}$ is the indiscrete topology.) We call $\tau(\mathcal{F})$ the **topology generated** by \mathcal{F} .

In fact $(\text{Top}(X), \leq)$ is a **complete lattice**. We recall what this means:

- (i) There is a “top element” in $\text{Top}(X)$, i.e., a topology which is finer than any other topology on X : namely the discrete topology.
- (ii) There is a “bottom element” in $\text{Top}(X)$, i.e., a topology which is coarser than any other topology on X : namely the indiscrete topology.
- (iii \wedge) If $\mathcal{T} \subset \text{Top}(X)$ is any family of topologies on X , then the **meet** $\wedge \mathcal{T}$ (or **infimum**) exists in $\text{Top}(X)$: there is a unique topology $\tau_{\wedge \mathcal{T}}$ on X such that for any $\tau \in \text{Top}(X)$, $\tau \leq \tau_{\wedge \mathcal{T}}$ iff $\tau \leq T$ for all $T \in \mathcal{T}$: namely we just take the intersection $\bigcap_{T \in \mathcal{T}} T$, as in Exercise X.X above.
- (iii \vee) If $\mathcal{T} \subset \text{Top}(X)$ is any family of topologies on X , then the **join** $\vee \mathcal{T}$ (or **supremum**) exists in $\text{Top}(X)$: there is a unique topology $\tau_{\vee \mathcal{T}}$ such that for any $\tau \in \text{Top}(X)$, $\tau \geq \tau_{\vee \mathcal{T}}$ iff $\tau \geq T$ for all $T \in \mathcal{T}$: we first take $\mathcal{F}(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} T$ and then $\vee \mathcal{T} = \tau(\mathcal{F})$ is the intersection of all topologies containing \mathcal{F} .

Let us now look a bit more carefully at the structure of the topology $\tau(\mathcal{F})$ generated by an arbitrary family \mathcal{F} of subsets of X . The above description is a “top down” or an “extinsic” construction. Such situations occur frequently in mathematics, and it is also useful (maybe more useful) to have a complementary “bottom up” or

¹One sometimes also says, especially in functional analysis, that τ_1 is **weaker** than τ_2 and that τ_2 is **stronger** than τ_1 . Unfortunately some of the older literature uses the terms “weaker” and “stronger” in exactly the opposite way! So the coarser/finer terminology is preferred.

“intrinsic construction”.

By way of comparison, if G is a group and S is a subset of G , then there is a notion of the subgroup $H(S)$ generated by S . The “extrinsic” construction is again just $\bigcap_{H \supset S} H$, the intersection over all subgroups containing S . But there is also a well-known “intrinsic construction” of $H(S)$: namely, as the collection of all group elements of the form $x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$, where $x_i \in S$ and $\epsilon_i \in \pm 1$. In some sense, this “bottom up” construction is a two-step process: starting with the set S , we first replace S by $S \cup S^{-1}$, and second we pass to all words (including the empty word!) in $S \cup S^{-1}$.

In general, we may not be so lucky. If X is a set and \mathcal{F} is a family of subsets of X , in order to form the σ -algebra generated by \mathcal{F} , extrinsically we again just take the intersection over all σ -algebras on X containing \mathcal{F} (in particular there is always 2^X , so this intersection is nonempty). Sometimes one needs the intrinsic description, but this is usually avoided in first courses on measure theory because it is very complicated: one alternates the processes of passing to countable unions and adjoining complements, but in general one must do this uncountably many times, necessitating a transfinite induction!²

Luckily, the case of topological spaces is much more like that of groups than that of σ -algebras. Namely, starting with $\mathcal{F} \subset 2^X$, we first form \mathcal{F}_1 which consists of all finite intersections of elements of \mathcal{F} (employing, as usual, the convention that the empty intersection is all of X). We then form \mathcal{F}_2 , which consists of all arbitrary unions of elements of \mathcal{F}_1 (employing, as usual, the fact that the empty union is \emptyset). Clearly \mathcal{F}_2 contains \emptyset and X and is stable under arbitrary unions. In fact it is also stable under finite intersections, since for any two families $\{Y_i\}_{i \in I}$, $\{Z_j\}_{j \in J}$ of elements of \mathcal{F}_1 ,

$$\bigcup_i Y_i \cap \bigcup_j Z_j = \bigcup_{i,j} Y_i \cap Z_j,$$

and for all i and j $Y_i \cap Z_j \in \mathcal{F}_1$ since \mathcal{F}_1 was constructed to be closed under finite intersections. So we are done in two steps: $\mathcal{F}_2 = \tau(\mathcal{F})$ is the topology generated by \mathcal{F} .

Example X.X: Let X be any nonempty set. If $\mathcal{F} = \emptyset$, then $\tau(\mathcal{F})$ is the trivial topology. If $\mathcal{F} = \{\{x\} \mid x \in X\}$, $\tau(\mathcal{F})$ is the discrete topology. More generally, let S be any subset of X and $\mathcal{F}(S) = \{\{x\} \mid x \in S\}$, then $\tau(S) := \tau(\mathcal{F}(S))$ is a topology whose open points are precisely the elements of S , so this is a different topology for each $S \in 2^X$.

²To read more about this, the keyword is **Borel hierarchy**.