§2: THE NOTION OF A TOPOLOGICAL SPACE

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Part of the rigorization of analysis in the 19th century was the realization that notions like continuity of functions and convergence of sequences (e.g. $f : \mathbb{R}^n \to \mathbb{R}^m$) were most naturally formulated by paying close attention to the mapping properties between subsets U of the domain and codomain with the property that when $x \in U$, there exists $\epsilon > 0$ such that $||y - x|| < \epsilon$ implies $y \in U$. Such sets are called **open**. In the early twentieth century it was realized that many of the constructions formerly regarded as "analytic" in nature could be carried out in a very general context of sets and maps between them, provided only that the sets are endowed with a distinguished family of subsets, decreed to be open, and satisfying some very mild axioms. This led to the notion of an abstract topological space, as follows.

Definition: Let X be a set. A **topology** on X is a family $\tau = \{U_i\}_{i \in I}$ is a of subsets of X satisfying the following axioms:

(T1) $\emptyset,\ X \in \tau$.
(T2) $U_1,\ U_2 \in \tau \implies U_1 \cap U_2 \in \tau$.
(T3) For any subset $J \subset I$, $\bigcup_{i \in J} U_i \in \tau$.

It is pleasant to also be able to refer to axioms by a descriptive name. So instead of "Axiom (T2)" one generally speaks of a family $\tau \subset 2^X$ being closed under binary intersections. Similarly, instead of "Axiom (T3)", one says that the family τ is closed under arbitrary unions.

Remark: Consider the following variant of (T2):

(T2') For any finite subset $J \subset I$, $\bigcap_{i \in J} U_i \in \tau$.

Evidently (T2') $\implies$ (T2), and at first glance the converse seems to hold. This is almost, but not quite, true: the point of (T2') is to also allow the empty intersection, which is (by convention, really) defined as $\bigcap_{Y \in \emptyset} Y = X$. Since we also have (more transparently) $\bigcup_{Y \in \emptyset} Y = \emptyset$, it follows that (T2') + (T3) $\implies$ (T1). None of this is of any particular importance, but the reader should be aware of it because this alternative ("more efficient") axiomatization appears in some texts (e.g. Bourbaki's *Topologie Générale*).

A **topological space** (X, τ) consists of a set X and a topology τ on X. The elements of τ are called **open sets**.

If (X, τ_X) and (Y, τ_Y) are topological spaces, a map $f : X \to Y$ is *continuous* if for all $V \in \tau_Y$, $f^{-1}(V) \in \tau_X$. A function $f : X \to Y$ between topological spaces is a **homeomorphism** if it is bijective, continuous, and has a continuous inverse. A function f is **open** if for all $U \in \tau_X$, $f(U) \in \tau_Y$.

Exercise X.X: For a function $f : X \to Y$ between topological spaces (X, τ_X) and (Y, τ_Y), show that TFAE:
(i) f is a homeomorphism.
(ii) f is bijective and for all $V \subset Y$, $V \in \tau_Y \iff f^{-1}(V) \in \tau_X$.
(iii) f is bijective and for all $U \subset X$, $U \in \tau_X \iff f(U) \in \tau_Y$.
(iv) f is bijective, continuous and open.

Tournant dangereuse: A continuous bijection need not be a homeomorphism!

Exercise: Let (X, τ_X), (Y, τ_Y), $(Z, \tau_Z$ be topological spaces, and $f : X \to Y$, $g : Y \to Z$ be continuous functions. Show that $g \circ f : X \to Z$ is a continuous function.

Thus we get a category **Top** whose objects are the topological spaces and whose morphisms are the continuous functions between them. Our definition of homeomorphism is chosen so as to coincide with the notion of isomorphism in the categorical sense.

Exercise X.X: Let (X, d) be a metric space. Show that the open subsets form a topology, which we say is **induced from** d.

We will say that a topological space (X, τ) is **metrizable** if there is some metric d on X such that τ is induced from d.

It turns out that there is a simple and useful definition of arbitrary products of topological spaces. However, an uncountable product of metrizable spaces (each with at least two points) is not metrizable. Thus the class of topological spaces up to homeomorphism is more general than the class of metric spaces up to homeomorphism.

On the other hand, there are times when having a metric is more convenient than just a topology. For this and other reasons it is of interest to have sufficient (or ideally, necessary and sufficient) conditions for the metrizability of a topological space. This is in fact one of the main problems in general topology and will be addressed later.

The following is a simple example of a property which is possessed by all metrizable spaces but not by all topological spaces.

Definition: A topological space X is **Hausdorff** if given distinct points x, y in X, there exist open sets $U \ni x$, $V \ni y$ such that $U \cap Y = \emptyset$.

Exercise X.X: Show that a metrizable topology is Hausdorff.

For a set X, $\tau = \{\emptyset, X\}$ is a topology on X, called the **indiscrete topology** (and also the **trivial topology**). If X has more than one element, this topology is not Hausdorff.

For a set X, $\tau = 2^X$, the collection of *all* subsets of X, forms a topology, called the **discrete** topology.

The discrete and indiscrete topologies coincide iff X has at most one element. Otherwise they are distinct and indeed give rise to non-homeomorphic spaces.

Exercise X.X: Show that a topological space is discrete iff for all $x \in X$, $\{x\}$ is open.

A discrete topological space is metrizable: on any set X consider the *discrete metric* $d(x,y) = 1-\delta_{x,y}$, i.e., is 1 if $x \neq y$ and is 0 if $x = y$. With this metric $B(x, \frac{1}{2}) = \{x\}$, so the discrete metric induces the discrete topology.

Exercise X.X: Suppose X is a finite topological space (i.e., X is a finite set, which then forces $\tau \subset 2^X$ to be finite as well). If X is Hausdorff, then it is discrete. In particular, finite metric spaces are discrete.

Example X.X (DVR topology): Consider the two element set $X = \{\circ, \bullet\}$. We take $\tau = \{\emptyset, \{\circ\}, X\}$. This gives a topology on X in which the point $\circ$ is open but the point $\bullet$ is not, so X is finite and nondiscrete, hence nonmetrizable.

Exercise X.X: Let X a set.
a) Show that, up to homeomorphism, there are precisely 3 topologies on a two-element set: the trivial topology, the discrete topology, and the DVR topology.
c) For $n \in \mathbb{Z}^+$, let $I(n)$ denote the number of isomorphism classes of topologies on $\{1, \ldots, n\}$. Show that $\lim_{n\to\infty} I(n) = \infty$. (Note that only one of these topologies, the discrete topology, is metrizable.)
d)* Can you describe the asymptotics of $I(n)$, or even give reasonable lower and/or upper bounds?[1]

Example (cofinite topology): Let X be an infinite set, and let τ consist of $\emptyset$ together with subsets whose *complement* is finite (or, for short, "cofinite subsets"). This is easily seen to form a topology, in which any two nonempty open sets intersect[2], hence a non-Hausdorff topology.

Example (**Moore plane**): Let X be the subset of $\mathbb{R}^2$ consisting of pairs (x, y) with $y \geq 0$, endowed with the following "exotic" topology: a subset U of X is open if: whenever it contains a point $P = (x, y)$ with $y > 0$ it contains some open Euclidean disk $B(P, \epsilon)$; and whenever it contains a point $P = (x, 0)$ it contains $P \cup B((x, \epsilon), \epsilon)$ for some $\epsilon > 0$, i.e., an open disk in the upper-half plane tangent to the x-axis at P. The Moore plane satisfies several properties shared by all metrizable spaces – it is **first countable** and **Tychonoff** – but not the property

[1] This question has received a lot of attention but is, to the best of my knowledge, open in general.

[2] When we say that two subsets intersect, we mean of course that their intersection is nonempty.

of **normality**. More on these properties later, of course.

Exercise X.X: Let (X, τ) and (Y, τ') be topological spaces and $f : X \to Y$ be a map between them. Show that f is continuous iff for all closed $A \subset Y$, $f^{-1}(A)$ is closed in X.

One might well ask: why not just require that topological spaces satisfy the Hausdorff axiom? It turns out that, in some areas of mathematics, there arise naturally spaces which do not satisfy this axiom. The following is probably the most important example:

Exercise X.X (Zariski topology): Let R be a commutative ring, and let $\operatorname{Spec} R$ be the set of prime ideals of R. For any subset S of R (including $\emptyset$, let $C(S)$ be the set of prime ideals containing S.
a) Show that $C(S_1) \cup C(S_2) = C(S_1 \cap S_2)$.
b) Show that, for any collection $\{S_i\}_{i \in I}$ of subsets of R, $\bigcap_i C(S_i) = C(\bigcup_i S_i)$.
c) Note that $C(\emptyset) = \operatorname{Spec} R$, $C(R) = \emptyset$.
Thus the $C(S)$'s form the closed sets for a topology, called the Zariski topology on $\operatorname{Spec} R$.
d) If $\varphi : R \to R'$ is a homomorphism of commutative rings, show that $\varphi^* : \operatorname{Spec} R' \to \operatorname{Spec} R$, $P \mapsto \varphi^{-1}(P)$ is a continuous map.
e) Let $\operatorname{rad}(R)$ be the radical of R. Show that the natural map $\operatorname{Spec}(R/\operatorname{rad}(R)) \to \operatorname{Spec}(R)$ is a homeomorphism.
f) Let R be a discrete valuation ring. Show that $\operatorname{Spec} R$ is the topological space of Example X.X above.
g) Let k be an algebraically closed field and $R = k[t]$. Show that $\operatorname{Spec}(R)$ can, as a topological space, be identified with k itself with the cofinite topology.

Exercise X.X: For a topological space X, TFAE:
(a) Given any ordered pair (x, y) of distinct points of X, there exists an open set U containing x but not y.
(b) For all points x in X, $\{x\}$ is a closed subset.
We will call a space satisfying these equivalent conditions **separated**. (More traditional names are $\mathbf{T}_1$ and **Fréchet**.)

Exercise: Show that any Hausdorff space is separated. Give a counterexample to show that the converse does not hold. (Hint: we have already met such spaces.)

Exercise X.X: Say that a point η in a topological space X is a *generic point* if the only closed subset containing η is X itself.
a) Show that X has a generic point iff it is not the union of two proper closed subsets; this property is called *irreducibility*.
b) Show that an irreducible (T1) space consists of a single point.
c) If R is a commutative ring, show that $\operatorname{Spec} R$ is irreducible iff $R/\operatorname{rad}(R)$ is an integral domain.