§1: FROM METRIC SPACES TO TOPOLOGICAL SPACES

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We assume that the reader has a good working familiarity with the notion of a metric space, but to fix terminology and for the sake of comparison, let us begin with a rapid review. Proofs are generally omitted; an element of the intended audience will, for the most part, know them already or be able to supply them for herself.

A **metric** on a set X is a function $d : X \times X \to [0, \infty)$ satisfying:

(M1) $d(x, y) = 0 \iff x = y$.
(M2) For all $x,\ y \in X$, $d(x, y) = d(y, x)$.
(M3) For all $x,\ y,\ z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

A **metric space** is a pair (X, d) consisting of a set X and a metric d on X. By the usual abuse of notation, when only one metric on X is under discussion we will typically refer to "the metric space X."

For metric spaces (X, d_X) and (Y, d_Y) a map $f : X \to Y$ is said to be an **isometric embedding** if for all $x_1,\ x_2 \in X$, $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$. Note that, using (MS1), this forces f to be injective. An **isometry** is an isometric embedding which is moreover surjective; then the set-theoretic inverse f^{-1} is also an isometry. Thus isometry is the natural notion of **isomorphism** of metric spaces.

Example X.X: For a positive integer d, and a real number $p \geq 1$, the function $d_p : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ by

$$d_p((x_1, \ldots, x_d), (y_1, \ldots, y_d)) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

is a metric, as is

$$d_\infty((x_1, \ldots, x_d), (y_1, \ldots, y_d)) = \max_i |x_i - y_i|.$$

That (M3) holds for d_p is very standard (but nontrivial) fact known as Minkowski's inequality.

Exercise X.X:
a) If $d = 1$, show that all these metrics are the same.
b) Show that $\lim_{p \to \infty} d_p = d_\infty$.

Although all these metrics are important in analysis, the metric d_2 is called "Euclidean" and plays a distinguished role in elementary geometry.

Exercise X.X: If (X, d) is a metric space and $Y \subset X$ is any subset, then restricting

d to $Y \times Y$ endows Y with the structure of a metric space.

In particular, in high school geometry, when they said that two subsets Y_1, Y_2 of $\mathbb{R}^d$ were congruent, what was actually meant was that they are isometric as submetric spaces of $(\mathbb{R}^d, d_2)$.

In general, the study of properties of metric spaces is an important branch of geometry. However, metric spaces also show up in calculus and analysis via the following notion:

Let X and Y be metric spaces, and $x \in X$. A mapping $f : X \to Y$ is **continuous at x** if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, x') < \delta$, $d(f(x), f(x')) < \epsilon$. A mapping is said to be **continuous** if it is continuous at all points x in X.

We can recast the definition of continuity in slightly softer language, as follows: for x in X and $\alpha > 0$, define the **open ball**

$$B(x, \alpha) = \{x' \in X \mid d(x', x) < \alpha\}.$$

Then continuity at $x \in X$ simply means that for all $\epsilon > 0$, there is some δ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$.

Exercise X.X: In $\mathbb{R}^2$ equipped with the metric d_p (for $1 \leq p \leq \infty$), sketch $B((0, 0), 1)$.

Although each of these balls looks slightly different – in the limiting cases our "balls" have what appear to our Euclidean eyes to be sharp corners – it is nevertheless the case that for any $p, p' \in [1, \infty)$, any $x \in \mathbb{R}^d$ and any $\alpha > 0$, there exists $\alpha > 0$ such that $B_p(x, \alpha) \subset B_{p'}(x, \alpha')$.

Two metrics d_1, d_2 on the same set X are said to be **equivalent** if for all $x \in X$ and all $\alpha > 0$, there exist $\beta, \gamma > 0$ such that

$$B_{d_2}(x, \beta) \subset B_{d_1}(x, \alpha) \subset B_{d_2}(x, \gamma).$$

Proposition 1. *Let (Y, d_Y) be a metric space, and $f : X \to Y$ a function. If d_1 and d_2 are equivalent metrics on X, then f is continuous with respect to d_1 iff it is continuous with respect to d_2.*

In particular this holds for the metrics d_p on $\mathbb{R}^d$: our notion of continuous function is independent of the choice of p.

It turns out that in analysis, one is almost always interested in continuous functions and not in isometries, so that metrics only "count" up to equivalence.

Of course, in working with equivalence classes of metrics we still need to supply at least one metric, and there are places where this is awkward. For instance, given metric spaces (X_1, d_1), (X_2, d_2), what is a reasonable metric to put on the product set $X_1 \times X_2$? We have already seen that there is no canonical answer, since taking $X_1 = X_2 = \mathbb{R}$ we have already endowed the product $\mathbb{R}^2$ with a different metric for each $p \in [1, \infty]$. Perhaps you think d_2 is the most natural: it might be, for geometry. But $d_\infty = \max(d_1, d_2)$ could be easier to work with and is in some respects nicer: if, for instance, d_1 and d_2 take values in some subset $S \subset [0, \infty)$ – e.g. if they are integral or rational-valued – then so does d_∞, whereas d_2, whose construction involves extracting a squareroot, does not. Again though, when it comes to continuous functions on $X_1 \times X_2$ it does not matter which we pick, but we are in the slightly awkward situation of having to pick *something.* One can get very far in modern mathematics by employing the principle of "minimizing noncanonical choices"[1]; here the awkwardness hints at the existence (or at least, the hope of the existence) of some more fundamental structure on $X_1 \times X_2$ which does not require us to make a choice.

As another example, what if we have a sequence of metric spaces (X_i, d_i) and want to put a metric on the product $\prod_i X_i$? Here we cannot take any of the d_p metrics because the corresponding expressions need not be finite. There is a way out of this, but it has the feel of a parlor trick:

Exercise: a) For any metric space (X, d), consider the *associated bounded metric* $d_b = \min(d, 1)$. Show that d_b and d are equivalent metrics.
b) Consider a sequence of metric spaces (X_n, d_n), and put $X = \prod_{i=1}^{\infty} X_n$. On $X \times X$ define $d(x, y) = \sum_{n=1}^{\infty} \frac{(d_n)_b(x_n, y_n)}{2^n}$. Show that d is a metric.

This construction gives us a way of speaking about continuous functions and convergent sequences on the Cartesian product X. But it is certainly not a very natural construction. Fundamentally, the problem is that given a collection $(X_\alpha, d_\alpha)_{\alpha \in I}$ of metric spaces, no natural metric presents itself on the Cartesian product. In fact, if the indexing set I is uncountable there is – assuming each X_α has at least two points – no way to endow the Cartesian product with a useful metric, as we shall explain later in more detail.

The modern perspective is that our difficulties arise from a confusion of metric and topological properties. The topological approach begins as follows: for a metric space (X, d), a subset $U \subset X$ is *open* if for each $x \in U$ there exists some $\delta > 0$ such that $B(x, \delta) \subset U$.

Exercise X.X: Show that equivalent metrics give rise to the same open sets.

Proposition 2. *For a function $f : (X, d_x) \rightarrow (Y, d_Y)$, TFAE:*
a) f is continuous.
b) For each open set V in Y, $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is an open set in X.

In other words, continuous functions can be characterized in terms of their inverse images preserving the open sets.

This leads to the following bold idea: instead of prescribing a metric d on X, just give *a priori* a collection of subsets of X which we will decree to be "open," and define continuous functions to be those whose preimages preserve these "open" subsets.

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[1] This viewpoint is espoused with particular zealotry by my former thesis advisor, Barry Mazur.