Now the types of $F(z)$, $f(z)$ are attained only along the directions of strongest growth. Consequently, at the points at which $F(z)$ and $F_\nu(z)$ satisfy an inequality of the form (9), the functions $F_\mu(z)$ $(\mu \neq \nu)$ do not attain the type $h$ of $F(z)$. A similar property holds for $f(1/z)$ and the $f_\nu(1/z)$. It therefore follows that $|F(z)|$ and $|f(1/z)|$ attain large values at corresponding points along each identical direction of strongest growth, given by a formula similar to (11), although the densities of such sets of points will, in general, be different along different directions of strongest growth. As already noted $|\psi(e^{-z})-f(z)|$ is bounded in the neighbourhood of the point $z=0$ and this remark completes the proof of the theorem.

## *References.*


1. E. Lindelöf, *Acta Soc. Scient. Fennicae*, 31 (1902), 1–79.
2. A. J. Macintyre, *Proc. London Math. Soc.* (2), 45 (1939), 1–20.
3. A. J. Macintyre and R. Wilson, *Journal London Math Soc.*, 22 (1947), 298–304.
4. A. J. Macintyre and R. Wilson, *Proc. London Math. Soc.* (2), 47 (1940), 60–80.
5. G. Pólya, *Math. Zeitschrift*, 29 (1929), 549–640.
6. J. M. Whittaker and R. Wilson, *Journal London Math. Soc.*, 14 (1939), 202–208.


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# THE PRODUCT OF TWO NON-HOMOGENEOUS LINEAR FORMS

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Let $L(\xi, \eta) = a\xi + b\eta$, $M(\xi, \eta) = c\xi + d\eta$ be two homogeneous linear forms in $\xi$, $\eta$ with real coefficients and determinant $\Delta = ad - bc > 0$, and let $p$, $q$ be any two real numbers. A great many proofs have been given of the well-known

THEOREM. *Integer values of* $\xi$, $\eta$ *exist such that*

$$|(L+p)(M+q)| \leqslant \tfrac{1}{4}\Delta.$$

I give here a further proof.

We also call $\Delta$ the determinant of the non-homogeneous lattice given by $x = a\xi + b\eta + p$, $y = c\xi + d\eta + q$.

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LEMMA. *If $A$, $B$, $C$ are three points of a two-dimensional lattice (homogeneous or non-homogeneous), of determinant $\Delta$, such that the triangle $ABC$ contains no lattice point inside or on its boundary other than $A$, $B$, $C$, then its area is $\frac{1}{2}\Delta$.*

For complete the parallelogram $ABCD$. This contains no lattice point other than $A$, $B$, $C$, $D$ and so it generates the lattice. Thus its area is $\Delta$ and so the area of triangle $ABC$ is $\frac{1}{2}\Delta$.

*Proof of Theorem.* Consider the region $R$ defined by $|xy| \leqslant \frac{1}{4}\Delta$. If we can show that $R$ contains a point of every non-homogeneous lattice of determinant $\Delta$, then clearly integers $\xi$, $\eta$ exist such that

$$|(L+p)(M+q)| \leqslant \tfrac{1}{4}\Delta,$$

and the theorem is proved.

Suppose that a lattice $\Lambda$ of determinant $\Delta'$ has no point in $R$. There will be points of $\Lambda$ in each of the four regions outside $R$†, and we can find four lattice points, one in each of these regions, forming a quadrilateral which contains no lattice point inside or on its boundary except at the vertices. Dividing this into two triangles we see by the lemma that its area is $\Delta'$.

We now show that its area is greater than $\Delta$.

Let the coordinates of the vertices be $(\alpha_1, \beta_1)$, $(-\alpha_2, \beta_2)$, $(-\alpha_3, -\beta_3)$, $(\alpha_4, -\beta_4)$, where the $\alpha_i$, $\beta_i$ are all positive, and $\alpha_i\beta_i > \frac{1}{4}\Delta$, $i = 1, 2, 3, 4$. Then

$$\Delta' = \text{area of quadrilateral}$$

$$= \tfrac{1}{2}[(\alpha_1\beta_2+\alpha_2\beta_1)+(\alpha_2\beta_3+\alpha_3\beta_2)+(\alpha_3\beta_4+\alpha_4\beta_3)+(\alpha_4\beta_1+\alpha_1\beta_4)]$$

$$> \frac{\Delta}{8}\left[\left(\frac{\alpha_1}{\alpha_2}+\frac{\alpha_2}{\alpha_1}\right)+\left(\frac{\alpha_2}{\alpha_3}+\frac{\alpha_3}{\alpha_2}\right)+\left(\frac{\alpha_3}{\alpha_4}+\frac{\alpha_4}{\alpha_3}\right)+\left(\frac{\alpha_4}{\alpha_1}+\frac{\alpha_1}{\alpha_4}\right)\right].$$

Therefore $\Delta' > \Delta$.

Thus every non-homogeneous lattice of determinant less than or equal to $\Delta$ has at least one point in $R$.

Hence the theorem is proved.

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† Clearly there exist arbitrarily large integer values of $\xi$, $\eta$ for which $x$ and $y$ have given signs and $|xy| > \frac{1}{4}\Delta$.