SHIMURA CURVES LECTURE 5: THE ADELIC PERSPECTIVE

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Recall our running notation: $F/\mathbb{Q}$ is a totally real field of degree n, B/F is a totally indefinite quaternion algebra (and we have been allowing the split case $B = M_2(F)$).

At this point we have already discussed two important matters.

(1) the adelic perspective – i.e., how to view quotients of $\mathcal{H}^g$ by certain arithmetic congruence subgroups as spaces of double cosets for the adelic points of semisimple groups, and also how replacing the semisimple group by a reductive group (i.e., adding a center) and performing the corresponding adelic construction gives finite disjoint unions of the classical spcaes $\Gamma\backslash\mathcal{H}^g$.

(2) How to view $V(\mathcal{O}) = \Gamma(B, \mathcal{O})\backslash\mathcal{H}^g$ as a moduli space for certain pairs (A, ι), where A is a complex abelian variety of dimension $2g$ and $\iota : \mathcal{O} \hookrightarrow \operatorname{End}(A)$ is a **QM structure**. (Here $\mathcal{O}$ was assumed to be a maximal order of B. In fact we did not use the maximality in any way. However, later on subtleties will arise when considering non-maximal orders. Just as an example: if $\mathcal{O}$ is not maximal, then it is not clear that any of the abelian varieties we've constructed actually have endomorphism ring isomorphic to $\mathcal{O}$ rather than merely containing it.)

However, when we tried to show that $V(\mathcal{O})$ parameterized *all* $2n$-dimensional abelian varieties with $\mathcal{O}$-QM we did not succeed (nor should we have!). By analyzing the possible complex structures on $\Lambda \otimes \mathbb{C}$, where Λ is a projective $\mathcal{O}$-module of rank 1 (so as abelian group, $\Lambda \cong \mathbb{Z}^{4n}$) up to equivalence (namely, equivalence preserving the holomorphic and the $\mathcal{O}$-QM structures) we ended up with the space

$$W(\mathcal{O}) = \mathcal{O}^\times\backslash(\mathcal{H}^\pm)^n,$$

which could *a priori* be larger than $V(\mathcal{O})$: more precisely, we can see that it has $1 \leq N \leq 2^n$ connected components, each of which is isomorphic to $V(\mathcal{O})$. Later on we will prove the following result:

Theorem 1. *If F has narrow class number 1, and $\mathcal{O}$ is maximal, then $W(\mathcal{O}) = V(\mathcal{O})$.*

In particular this occurs when $F = \mathbb{Q}$, the case to which we should probably be devoting more concentrated attention.

In fact, let us look a little bit more closely at the case $F = \mathbb{Q}$. Here we are just claiming that some element of $\mathcal{O}^\times$ interchanges the upper and lower halfplanes; equivalently, there exists a unit of $\mathcal{O}$ of reduced norm -1. Now recall that the form $N : B \to \mathbb{Q}$, $x \mapsto N(x)$ is a quadratic form which is (upon extension to $\mathbb{R}$)

indefinite. As discussed in class, the algebraic 5 *Quadratic Forms Over Fields* – but the case where the theory of quadratic forms tells us that any totally indefinite quaternary quadratic form over a number field F is universal, i.e., the map (character!) $N : B^\times \rightarrow F^\times$ is surjective. So certainly there exists some element of B of norm -1. A bit of classical number theory gives the following:

Exercise 1: a) Show that for any totally indefinite quaternion algebra B/F, there exists $\alpha \in B$ with norm -1 and whose reduced trace is an algebraic integer.
b) Conclude that there exists some maximal order $\mathcal{O}$ containing an element of norm -1. (An element of B is *integral* if both its reduced norm and reduced trace are integers in F. Any integral element is contained in an integral ideal – i.e., a lattice of integral elements – and the left order of an integral ideal contains that ideal.)
c) When $F = \mathbb{Q}$ (and in fact, whenever F has narrow class number 1) any two maximal orders are conjugate. Conclude that under this hypothesis every maximal order has a unit of norm -1.
d)* Does there exist a maximal order in a totally indefinite quaternion algebra *without* an element of norm -1?

Coming back to the general discussion, the fact that we are getting disconnected moduli spaces strongly suggests that we should employ an adelic construction: it simultaneously gives us "the right number of connected components" automatically and, if we want to understand how many components we have and/or the relationship between the components, it points the way to the requisite classfield theory.

Let us simplify notation slightly by writing D for $(\mathcal{H}^\pm)^n$ and D^+ for $\mathcal{H}^n$ (the "totally upper" connected component of D.

Recall our adelic construction: we took $G = B^\times$, viewed as a linear algebraic group over $\mathbb{Q}$ and $K_f \subset G(A_f)$ a compact open subgroup of the finite adelic points. Let $T := R_{F/\mathbb{Q}}(\mathbb{G}_m)$, i.e., $F^\times$ viewed as an algebraic group over $\mathbb{Q}$. The reduced norm map gives a character $N : G \rightarrow T$, and we denote by G' the semisimple group which is the kernel. The group $N(K_f)$ is itself a compact open subgroup of $T(A_f)$ (i.e., the finite idele group over F), and the quotient

$$F^\times \backslash \{\pm 1\}^n \times T(A_f)/N(K_f)$$

is finite (and corresponds to an abelian extension of F). Here we let $b_1, \ldots, b_N$ be a set of double coset representatives and choose $a_1, \ldots, a_N \in G(A_f)$ such that $N(a_i) = b_i$). Then we saw that the double coset space

$$M(G, K_f) = G(\mathbb{Q}) \backslash D \times G(A_f)/K_f)$$

was isomorphic to

$$\coprod_{i=1}^{N} \Gamma_i \backslash D^+;$$

where $\Gamma_i = G'(\mathbb{Q}) \cap a_i K_f a_i^{-1}$.

The goal of this lecture will be to give a moduli interpretation to this double coset construction, and especially to understand how the choice of K_f corresponds to a level structure.

Here we follow Milne's article *Points on Shimura varieties mod p.*

Step 0: It is more convenient to work with an integral form of G, namely we take $G = \mathcal{O}^\times$ viewed as a group over $\mathbb{Z}$. What this really means is that for any commutative ring R whatsoever, we can plug in $G(R)$ and this means $(\mathcal{O} \otimes R)^\times$.

Step 1: We will construct the QM-abelian variety corresponding to the point $(\sqrt{-1}, \ldots, \sqrt{-1}, 1)$. For this, we start with V a free $\mathbb{Z}$-module of rank $4n$ with an $\mathcal{O}$-action. Recall the following lemma:

Lemma 2. *(Milne) There exists a unique nondegenerate alternating form ψ on $V(\mathbb{Q})$ such that*
(a) $\Psi(V, V) \subset \mathbb{Z}$.
(b) $\psi(ut, u) < 0$ *for all* $u \neq 0, u \in V(\mathbb{R})$.
(c) $\psi(bu, v) = \psi(u, b^* v)$ *for all* $u,\ v \in V(\mathbb{Q})$.
(d) ...

Note that for any $\mathbb{Z}$-algebra R, we may identify $G(R) = B(R)^\times$ with $\mathrm{Aut}_{\mathcal{O}\otimes R}(V(R))$ since any $\mathcal{O} \otimes R$ endomorphism of $V(R) = \mathcal{O} \otimes R$ is right multiplication by an element of $\mathcal{O} \times R$. Taking now $R = \mathbb{R}$, we define a homomorphism $h : \mathbb{C}^\times \to G(\mathbb{R}) = \sum_{i=1}^{n} GL_2(\mathbb{R})$ such that $h(i)$ is right multiplication by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ on each factor. (As we discussed last time, this is the complex structure corresponding to "our favorite point" $(\sqrt{-1}, \ldots, \sqrt{-1})$ in D^+. Note that there is something obviously silly going on here: $G'(R)$ acts transitively on D^+, so there is no instrinsic sense in which this point is distinguished. In some sense, our current construction is remedying this.) The above form ψ is a Riemann form on $(V(\mathbb{Z}), h)$, so that (as discussed last time), we get a polarized abelian variety together with a QM-structure $\iota : \mathcal{O} \hookrightarrow (A, \iota, \psi)$. However, in this case, the QM-structure determines the polarization up to a certain equivalence, so we do not need to include it in the construction.

On the other hand, we must now address the $K_f \subset G(A_f)$. By $T_f(A)$ we denote the full Tate module of A, so the inverse limit of $A[n]$; we have $T_f(A) = \prod_\ell T_\ell(A)$, where $T_\ell(A)$ is the usual ℓ-adic Tate module. On the other hand, for a uniformized abelian variety like our $A = V(\mathbb{R})/V(\mathbb{Z})$, we have that $V(\mathbb{Z}) \otimes \hat{\mathbb{Z}} = V(\hat{\mathbb{Z}})$ is naturally isomorphic to $T_f(A)$, so that $V_f(A) = T_f(A) \otimes \mathbb{Q}$ is isomorphic to $V(A_f)$. This means that $K_f \subset G(A_f) = \mathrm{Aut}\, V(A_f)$ acts by automorphisms on the Tate module (tensored with $\mathbb{Q}$).

Definition: Let $\phi_1,\ \phi_2 : T_f(A) \cong V(\hat{\mathbb{Z}})$ be two isomorphisms. They are said to be K_f-equivalent if $\phi_1 = k\phi_2$ for some $k \in K$.

An important special case: $K_f = K(n)$ is the kernel of the natural map $G(\hat{\mathbb{Z}} \to G(\mathbb{Z}/n\mathbb{Z})$. Then giving a $K(n)$-equivalence class of isomorphisms is giving an isomorphism from $A[n]$ to $V(\mathbb{Z}/n\mathbb{Z})$, i.e., a full level n structure. Recall that the adelic topology on $G(\hat{\mathbb{Z}})$ is such that the $K(n)$'s are cofinal in the compact open subgroups.

Step 2: Note that D is equal to the conjugacy class of h in $G(\mathbb{R})$, i.e., to the set of complex structures on $V(\mathbb{R})$ compatible with the QM-structure. Thus, taking K_∞ to be the centralizer of h in $G(\mathbb{R})$, we can also write

$$M(G, K_f) = G(\mathbb{Q})\backslash G(A)/K_\infty K_f.$$

Theorem 3. *There is a bijective correspondence between the points of $M(G, K_f)$ and the set of isomorphism classes of triples (A, ι, ϕ), where A is an abelian variety of dimension $2d$, $\iota : \mathcal{O} \times \mathrm{End}(A)$ is a QM structure and ϕ is a K_f-equivalence class of $\mathcal{O}$-isomorphisms $T_f(A) \cong V(\hat{\mathbb{Z}})$.*

Remark: This construction is one of a rather large class of similar examples. Others include:

(i) Siegel moduli space, with $G = GSp_{2g}$.
(ii) Hilbert moduli space, $G = R_{F/\mathbb{Q}}(GL_2)$.
(iii) CM points, $G = R_{K/\mathbb{Q}}(\mathbb{G}_m)$, where K is a CM field.

There are many more: given an algebra B equipped with a positive involution $*$,, a finite free B-module V^0, and a symplectic form $\psi : V^0 \times V^0 \to \mathbb{Q}$ which satisfies the adjunction condition $\psi(bu, v) = \psi(u, b^* v)$ for all $u, v \in V^0$, $b \in B$ (satisfying a tiny extra condition that we will not enter into here), one puts G to be the algebraic group of B-equivariant symplectic similitudes of V (so G is naturally a subgroup of a GSP and hence a matrix group), and G' to be the subgroup of matrices with determinant 1 and strictly preserving the symplectic form. Then one has an entirely analogous construction, giving abelian varieties of dimension $\frac{1}{2}\dim_{\mathbb{Q}} V$ with an injection $B \hookrightarrow \mathrm{End}^0(A)$.