It follows that $\Sigma |a_n| e^{-x\lambda_n}$ converges for $x > 0$. This completes the proof for this case.

In the case when

$$\underline{\lim}\, (\lambda_{n+1} - \lambda_n) \lambda_n^{-1} u_{n+1} \neq 0$$

so that, by (1),

$$\underline{\lim}\, (\lambda_{n+1} - \lambda_n) \lambda_n^{-1} u_n = 0,$$

the modifications required are obvious, and the argument presents no new difficulty.

COROLLARY. *If an (A, λ) method sums only series whose terms are bounded, then it sums only convergent series, and $\{\lambda_n\}$ satisfies Littlewood's high indices condition.*

For we may take $u_n = 1$. Then by Theorem A,

$$\frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} = O(1),$$

and this is equivalent to the high indices condition. The corollary now follows from the theorem of Hardy and Littlewood.

I should like to express my thanks to Dr. Bosanquet, who originated the problem, and to Dr. Kuttner, who suggested a substantial simplification in the proof.

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THE NUMBER OF LATTICE POINTS IN A STAR BODY

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1. Let K be a convex body in n-dimensional space, with the origin O as centre and having volume $V(K)$. Let Λ be a lattice with determinant $d(\Lambda)$. A well known theorem of Minkowski asserts that, if

$$V(K) > 2^n d(\Lambda),$$

then there is a pair of lattice points $\pm A$ of Λ in K. Van der Corput† has shown that, if m is a positive integer and

$$V(K) > m 2^n d(\Lambda),$$

then there are m distinct pairs $\pm A_1, \ldots, \pm A_m$ of points of Λ in K.

* Received 15 November, 1950; read 16 November, 1950.

† *Acta Arithmetica*, 2 (1936), 145–146.

Let S be a star set with the origin O as centre, and let $\Delta(S)$ be the lower bound of the determinants $d(\Lambda)$ of the lattices Λ with no point other than O in S. Then, corresponding to the result of Minkowski stated above, we have the trivial result that, if Λ is any lattice and

$$\Delta(S) > d(\Lambda),$$

then there is a pair of points $\pm A$ of Λ in S. These results suggest that corresponding to van der Corput's result one might make the following conjecture.

CONJECTURE. *Let m be a positive integer, let S be a star set with O as centre, and let Λ be a lattice. Then, if*

$$\Delta(S) > md(\Lambda),$$

there are m distinct pairs $\pm A_1, \ldots, \pm A_m$ of points of Λ in S.

While I have been able to prove this conjecture when m is of certain special forms, I have not been able to prove it in general. In this note I confine my attention to the case when m is a prime number. In this case we prove the following slightly stronger result.

THEOREM 1. *Let p be a prime, let S be a star set with O as centre and let Λ be a lattice. Then, if*

$$\Delta(S) > pd(\Lambda), \tag{1}$$

either (i) *there is a primitive point A_1 of Λ such that the points $\pm A_1, \pm 2A_1, \ldots, \pm pA_1$ are in S, or* (ii) *there are $p+1$ distinct pairs $\pm A_1, \ldots, \pm A_{p+1}$ of primitive points of Λ in S.*

In Section 3 we use this theorem to give a simple proof of the following theorem, stated by Minkowski* and proved by Hlawka†.

THEOREM 2. *Let S be a bounded star set, with O as centre, having Jordan measure $V(S)$ satisfying*

$$V(S) < 2\zeta(n). \qquad \left(\zeta(n) = \sum_{k=1}^{\infty} k^{-n}\right) \tag{2}$$

Then there exists a lattice Λ, with determinant 1, having no point other than O in S.

2. Before we prove Theorem 1 it is convenient to prove a lemma.

LEMMA 1. *Let p be any prime number and let $l_1, \ldots, l_n$ be integers, not all divisible by p. Then the set Λ of all points $U = (u_1, \ldots, u_n)$ with integral*

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* H. Minkowski, *Gesammelte Abhandlungen* (Leipzig, 1911), vol. 1, 265, 270 and 277.

† E. Hlawka, *Math. Zeit.*, 49 (1944), 285–312.

coordinates $u_1, \ldots, u_n$ *satisfying*

$$l_1 u_1 + \ldots + l_n u_n \equiv 0 \pmod{p} \tag{3}$$

is a lattice with determinant p.

Proof. We may suppose without loss of generality that l_1 is not divisible by p. Choose integers r and s such that $rl_1 + sp = 1$. Then r is not divisible by p, and a point $(u_1, \ldots, u_n)$ with integral coordinates satisfies (3) if and only if

$$u_1 + rl_2 u_2 + \ldots + rl_n u_n \equiv 0 \pmod{p}.$$

Now it is clear that Λ is the lattice with determinant p generated by the points

$$(p, 0, \ldots, 0),\ (-rl_2, 1, \ldots, 0),\ \ldots,\ (-rl_n, 0, \ldots, 1).$$

Proof of Theorem 1. Suppose that (1) is satisfied. Let $\pm A_1, \ldots, \pm A_m$ be the pairs of primitive points of Λ in S. We suppose that $m < p+1$ and that none of the points $\pm pA_1, \ldots, \pm pA_m$ are in S, and we will obtain a contradiction. We may suppose without real loss of generality that Λ is the lattice of points with integral coordinates. Then (1) takes the form

$$\Delta(S) > p. \tag{4}$$

Let $A_\mu = (a_1^{(\mu)}, \ldots, a_n^{(\mu)})$ for $\mu = 1, \ldots, m$. Then, by the lemma, the points $L = (l_1, \ldots, l_n)$ of Λ satisfying

$$l_1 a_1^{(\mu)} + \ldots + l_n a_n^{(\mu)} \equiv 0 \pmod{p}$$

form a lattice Λ_μ of determinant p, for $\mu = 1, \ldots, m$. Further the number of points L of Λ_μ other than O satisfying

$$0 \leqslant l_1 < p,\ \ldots,\ 0 \leqslant l_n < p, \tag{5}$$

is $p^{n-1} - 1$. Thus the total number of points L other than O, belonging to at least one of the lattices $\Lambda_1, \ldots, \Lambda_m$ and satisfying (5), is at most $(p^{n-1} - 1)m$, which is less than $p^n - 1$, as $m \leqslant p$. But the total number of points L of Λ other than O, satisfying (5), is $p^n - 1$. So we can choose a point L of Λ, other than O, which satisfies (5), but does not belong to any of the lattices $\Lambda_1, \ldots, \Lambda_m$. Clearly not all the coordinates $l_1, \ldots, l_n$ of this point L are divisible by p. Hence, by the lemma, the set of all points $U = (u_1, \ldots, u_n)$ of Λ, satisfying

$$l_1 u_1 + \ldots + l_n u_n \equiv 0 \pmod{p},$$

is a lattice Λ_0 with determinant p.

Now it follows from (4) that there is a point A of Λ_0 other than O in S. As S is a symmetric star set we have $A = rA_\mu$, for some integers r, μ with

$r \neq 0$ and $1 \leqslant \mu \leqslant m$. Since the points $\pm pA_\mu$ are not in S we have $0 < |r| < p$. As rA_μ is a point of Λ_0 we have

$$rl_1 a_1^{(\mu)} + \ldots + rl_n a_n^{(\mu)} \equiv 0 \pmod{p}.$$

Hence, r being relatively prime to p,

$$l_1 a_1^{(\mu)} + \ldots + l_n a_n^{(\mu)} \equiv 0 \pmod{p},$$

and L is a point of Λ_μ. This gives a contradiction and completes the proof of the theorem.

3. *Proof of Theorem* 2. Let Λ_ϵ be the lattice of points whose coordinates are integral multiples of ϵ, for any fixed positive ϵ. Let $m(\epsilon)$ be the number of pairs of primitive points of Λ_ϵ in S. Let $p = p(\epsilon)$ be the largest prime number satisfying

$$\epsilon^n p(\epsilon) < \Delta(S).$$

Then, since the ratio of consecutive primes tends to 1,

$$\lim_{\epsilon \to +0} \epsilon^n p(\epsilon) = \Delta(S). \tag{6}$$

As S is bounded, we can choose R so large that S is contained in the cube

$$|x_1| < R, \ \ldots, \ |x_n| < R. \tag{7}$$

Choose $\epsilon_0 > 0$ so small that $\epsilon p(\epsilon) > R$ when $0 < \epsilon < \epsilon_0$. Then, if $0 < \epsilon < \epsilon_0$ and A is any point of Λ_ϵ other than O, the point pA is not in the cube (7) and so is not in S. Hence by Theorem 1 we must have $m(\epsilon) > p(\epsilon)$ when $0 < \epsilon < \epsilon_0$. Thus, using (6), we have

$$\varliminf_{\epsilon \to +0} \epsilon^n m(\epsilon) \geqslant \lim_{\epsilon \to +0} \epsilon^n p(\epsilon) = \Delta(S). \tag{8}$$

But since S is Jordan measurable it is easy to prove* that

$$\lim_{\epsilon \to +0} \epsilon^n m(\epsilon) = \frac{V(S)}{2\zeta(n)}. \tag{9}$$

Combining (8) and (9), we have

$$\Delta(S) \leqslant \frac{V(S)}{2\zeta(n)}.$$

The required result follows from this inequality and the definition of $\Delta(S)$.

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* For a proof of this, see my paper in *Annals of Math.*, 48 (1947), 994–1002 (998).