$p \geqslant 1$, $h = 1{\cdot}6438$ for $p \geqslant 2{\cdot}35$. Similarly, in case (ii) we take $v < \frac{h-1}{h+1}$, with the values of h used in the first part of this proof.

The inequalities (5) and (6) now follow on verifying (9) to (12) for $1 \leqslant p \leqslant 1{\cdot}25$, $2{\cdot}35 \leqslant p \leqslant 2{\cdot}8$, $1{\cdot}65 \leqslant p \leqslant 2$, and $2 \leqslant p \leqslant 2{\cdot}15$, respectively. Since in each case we have a monotone increasing function of p on each side of the inequality, we may follow the same pedestrian method as we have used to prove (3) and (4); it takes about four to six steps in each case. More sophisticated methods can be devised, but little if any labour is saved.

I am indebted to the referee for some suggestions which have helped to simplify the proofs.

References.

1. Minkowski, H., *Diophantische Approximationen* (Berlin, Teubner, 1907), 51–58.
2. Davis, C. S., "Note on a conjecture by Minkowski", *Journal London Math. Soc.*, 23 (1948), 172–175.

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ON POSITIVE DEFINITE QUADRATIC FORMS

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Let $f(x_1, x_2, \ldots, x_n)$ be a positive definite quadratic form in n variables, with real coefficients. For reasons of homogeneity we can restrict ourselves to forms of determinant 1. Various upper bounds are known, depending only on n, for the minimum $M(f)$ of f for integral $x_1, x_2, \ldots, x_n$, not all zero, and it is known that there are extreme forms for which $M(f)$ attains its maximum.

$M(f)$ can be interpreted as the minimum of any diagonal coefficient of any form equivalent to f. In this paper we consider a generalization of $M(f)$ suggested by this interpretation. If r is one of $1, 2, \ldots, n-1$, we denote by $M_r(f)$ the lower bound of any principal minor of order r of any form equivalent to f.

The author is indebted to the referee for suggestions which have shortened considerably the proofs of Theorems 1 to 4.

THEOREM 1. *For given f, the lower bound $M_r(f)$ is attained.*

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† Received 6 May, 1952; read 15 May, 1952; revised 24 October, 1952.

Proof. Express f as $\xi_1^2+\xi_2^2+\ldots+\xi_n^2$, where $\xi_1, \xi_2, \ldots, \xi_n$ are linear forms of determinant 1 in $x_1, x_2, \ldots, x_n$. The points $(\xi_1, \xi_2, \ldots, \xi_n)$ which correspond to integral values of $x_1, x_2, \ldots, x_n$ constitute a lattice of determinant 1. The operation of equating to zero $n-r$ of the x's (or of other variables related to the x's by a unimodular substitution) is the operation of selecting an r-dimensional sublattice of Λ. Every principal minor of order r of a form equivalent to f is the square of the determinant of an r-dimensional sublattice of Λ. It remains to prove that there is only a finite number of distinct r-dimensional sublattices of Λ whose determinants are bounded by any given number. This is immediate, since such a sublattice of the given lattice Λ can always be generated by r points in a bounded portion of space, and there is only a finite number of possibilities for these points.

THEOREM 2. *If F is the form adjoint to f, then $M_r(f) = M_{n-r}(F)$.*

Proof. If Δ_r is any minor of f of order r, and Δ'_{n-r} is the complementary minor of F, of order $n-r$, it is well known that $\Delta_r = \Delta'_{n-r}$. The result follows.

THEOREM 3. *$M_r(f)$ is bounded above for all forms of determinant* 1, *and attains its upper bound.*

Proof. Every form f corresponds to a lattice Λ, as explained in the proof of Theorem 1. It is well known that every lattice can be generated by points $P_1, P_2, \ldots, P_n$ satisfying

$$|P_1| \leqslant |P_2| \leqslant \ldots \leqslant |P_n|, \quad |P_1| \, . \, |P_2| \, . \, \ldots \, . \, |P_n| \leqslant \lambda_n, \tag{1}$$

where λ_n depends only on n. Here $|P|$ denotes the distance of P from the origin. The determinant of the sublattice generated by $P_1, \ldots, P_n$ does not exceed

$$|P_1| \, . \, |P_2| \, . \, \ldots \, . \, |P_r| \leqslant \lambda_n^{r/n}.$$

It follows that $M_r(f) \leqslant (\lambda_n)^{2r/n}$.

Since $M_r(f) = 1$ when $f = x_1^2+x_2^2+\ldots+x_n^2$, we can restrict ourselves to forms for which $M_r(f) \geqslant 1$ when considering the upper bound of $M_r(f)$. If $P_1, \ldots, P_n$ is a basis, satisfying (1), for a lattice corresponding to such a form, then $|\det(P_1, P_2, \ldots, P_r)| \geqslant 1$, whence

$$|P_1| \, . \, |P_2| \, . \, \ldots \, . \, |P_r| \geqslant 1. \tag{2}$$

It follows from (1) and (2) that $|P_1|, \ldots, |P_n|$ are all bounded. Since the minimum determinant of an r-dimensional sublattice of the lattice generated by $P_1, \ldots, P_n$ is obviously a continuous function of $P_1, \ldots, P_n$, the conclusion follows.

We define $\gamma_{n,r}$ to be the maximum of $M_r(f)$ for all forms f of determinant 1. We have, by Theorem 2, $\gamma_{n,r} = \gamma_{n,n-r}$.

THEOREM 4. *If* $1 \leqslant m < r \leqslant n-1$, *then*

$$\gamma_{n,m} \leqslant \gamma_{r,m} (\gamma_{n,r})^{m/r}.$$

Proof. We can transform any form f in n variables into one for which the principal minor of order r is $M_r(f) \leqslant \gamma_{n,r}$. If we put the remaining $n-r$ variables equal to zero we obtain a form in r variables whose determinant is $M_r(f)$. Some form equivalent to this has a principal minor of order m which does not exceed $\gamma_{r,m}(\gamma_{n,r})^{m/r}$, and the result follows.

Theorem 4 is a generalization of Mordell's inequality† $\gamma_n = \gamma_{n-1}^{(n-1)/(n-2)}$, which may be deduced from it by putting $m = n-r = 1$.

The first number $\gamma_{n,r}$ for which $1 < r < n-1$ is $\gamma_{4,2}$. We prove

THEOREM 5. $\gamma_{4,2} = \frac{3}{2}$. *We have, when* $n = 4$, $M_2(f) < \gamma_{4,2} D^{1/2}$ *except when* f *is equivalent to* af^*, *where* a *is a constant, and*

$$f^*(x_1, x_2, x_3, x_4) = x_1^2+x_2^2+x_3^2+x_4^2+x_1 x_2+x_1 x_3+x_1 x_4+x_2 x_3+x_3 x_4.$$

Proof. From Theorem 4 we conclude that

$$\gamma_{4,2} \leqslant \gamma_{3,2}\gamma_{4,3}^{2/3} = 2^{2/3} = 1{\cdot}587\ldots,$$

since $\gamma_{3,2} = 2^{1/3}$, $\gamma_{4,3} = 2^{1/2}$. Thus Theorem 5 is an improvement on this result.

Let $f(x)$ be a positive definite form in n variables, where x denotes the point $(x_1, \ldots, x_n)$. It is convenient to express $f(x)$ in the following *canonical* form:

$$\begin{aligned} f(x) &= A_1(x_1+\alpha_{12} x_2+\ldots+\alpha_{1n} x_n)^2+A_2(x_2+\alpha_{23} x_3+\ldots+\alpha_{2n} x_n)^2 \\ &\qquad +\ldots+A_{n-1}(x_{n-1}+\alpha_{n-1,n} x_n)^2+A_n x_n^2 \\ &= A_1 X_1^2+A_2 X_2^2+\ldots+A_n X_n^2, \end{aligned}$$

say, where A_1 is the minimum of $f(x)$, A_2 is the minimum of $f(x)-A_1 X_1^2$, A_3 is the minimum of $f(x)-A_1 X_1^2-A_2 X_2^2$, etc., and where $-\frac{1}{2} < \alpha_{ij} \leqslant \frac{1}{2}$ $(1 \leqslant i < j \leqslant n)$. Hermite showed that in such a canonical form $A_{i+1} \geqslant \frac{3}{4}A_i$ $(1 \leqslant i \leqslant n-1)$, and Korkine and Zolotareff‡ proved that

$$A_{i+2} \geqslant \tfrac{2}{3}A_i \quad (1 \leqslant i \leqslant n-2). \tag{3}$$

We note that if $f(x)$ is expressed in canonical form then the quadratic form in $n-1$ variables obtained by putting $x_n = 0$ is also expressed in canonical

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† L. J. Mordell, "Observation on the minimum of a positive quadratic form in eight variables", *Journal London Math. Soc.*, 19 (1944), 1–6.

‡ A. Korkine and G. Zolotareff, "Sur les formes quadratiques", *Math. Annalen*, 6 (1873), 366–389.

form. It should also be noted that we do not claim that any canonical representation is unique.

We now take $n=4$ and suppose that f has a canonical representation

$$f(x) = A_1(x_1+\alpha_{12}x_2+\alpha_{13}x_3+\alpha_{14}x_4)^2+A_2(x_2+\alpha_{23}x_3+\alpha_{24}x_4)^2 \\ +A_3(x_3+\alpha_{34}x_4)^2+A_4x_4^2.$$

Then we have, by (3),

$$A_3 \geqslant \tfrac{2}{3}A_1, \quad A_4 \geqslant \tfrac{2}{3}A_2, \quad A_1A_2A_3A_4 = 1. \tag{4}$$

Hence $$M_2(f) \leqslant A_1A_2 \leqslant \tfrac{3}{2}(A_1A_2A_3A_4)^{1/2} =$$

which shows that $\gamma_{4,2} \leqslant \frac{3}{2}$.

Now $\gamma_{4,2}$ can equal $\frac{3}{2}$ only if $M_2(f)=\frac{3}{2}$ for some quaternary form f. For such a form we have

$$\tfrac{3}{2} = M_2(f) \leqslant A_1A_2 \leqslant \tfrac{3}{2}(A_1A_2A_3A_4)^{1/2} = \tfrac{3}{2}.$$

Hence $A_1A_2 = \frac{9}{4}A_3A_4$, and it follows from (4) that in every canonical representation of $f(x)$ we have

$$A_3 = \tfrac{2}{3}A_1, \quad A_4 = \tfrac{2}{3}A_2. \tag{5}$$

Now Korkine and Zolotareff have shown that if $A_3 = \frac{2}{3}A_1$ for any ternary form $g(y)$, then either

$$g(y) \sim g_1(y) = a_1\{(y_1+\tfrac{1}{2}y_2+\tfrac{1}{2}y_3)^2+\tfrac{3}{4}(y_2+\tfrac{1}{3}y_3)^2+\tfrac{2}{3}y_3^2\}$$
$$= a_1(y_1^2+y_2^2+y_3^2+y_1y_2+y_1y_3+y_2y_3)$$

or
$$g(y) \sim g_2(y) = a_2\{(y_1+\tfrac{1}{3}y_2-\tfrac{1}{3}y_3)^2+\tfrac{8}{9}(y_2+\tfrac{1}{2}y_3)^2+\tfrac{2}{3}y_3^2\}$$
$$= a_2(y_1^2+y_2^2+y_3^2+\tfrac{2}{3}y_1y_2-\tfrac{2}{3}y_1y_3+\tfrac{2}{3}y_2y_3).$$

Here the expressions given first for $g_1(y)$ and $g_2(y)$ are in canonical form. It follows that we can, by making a unimodular transformation if necessary, assume that $f(x)$ is of one of the two following canonical forms:

$$f(x) = A_1(x_1+\alpha x_2+\beta x_3+\gamma x_4)^2+A_2\{(x_2+\tfrac{1}{2}x_3+\tfrac{1}{2}x_4)^2+\tfrac{3}{4}(x_3+\tfrac{1}{3}x_4)^2+\tfrac{2}{3}x_4^2\}, \tag{6}$$

or

$$f(x) = A_1(x_1+\alpha x_2+\beta x_3+\gamma x_4)^2+A_2\{(x_2+\tfrac{1}{3}x_3-\tfrac{1}{3}x_4)^2+\tfrac{8}{9}(x_3+\tfrac{1}{2}x_4)^2+\tfrac{2}{3}x_4^2\}, \tag{7}$$

where $$-\tfrac{1}{2} < \alpha \leqslant \tfrac{1}{2}, \quad -\tfrac{1}{2} < \beta \leqslant \tfrac{1}{2}, \quad -\tfrac{1}{2} < \gamma \leqslant \tfrac{1}{2}. \tag{8}$$

We suppose, first of all, that $f(x)$ is of the form (6). Hence we have, by (5), $\frac{3}{4}A_2 = \frac{2}{3}A_1$; *i.e.* $A_2 = \frac{8}{9}A_1$, so that

$$f(x) = A_1\{(x_1+\alpha x_2+\beta x_3+\gamma x_4)^2+\tfrac{8}{9}(x_2^2+x_3^2+x_4^2+x_2x_3+x_3x_1+x_1x_2)\}.$$

Since A_1 is the minimum of f we deduce that $\alpha^2+\frac{8}{9} \geqslant 1$ by putting $x_1=x_3=x_4=0$, $x_2=1$. Similarly, if we put $x_2=-x_3=\pm 1$, $x_4=0$ and

$x_1 = 0$ or 1, we deduce that $(\alpha-\beta)^2+\frac{8}{9} \geqslant 1$ and that $(1-|\alpha-\beta|)^2+\frac{8}{9} \geqslant 1$. From this we obtain

$$|\alpha| \geqslant \tfrac{1}{3}, \quad \tfrac{1}{3} \leqslant |\alpha-\beta| \leqslant \tfrac{2}{3},$$

and we find, similarly,

$$|\beta| \geqslant \tfrac{1}{3}, \quad |\gamma| \geqslant \tfrac{1}{3}, \quad \tfrac{1}{3} \leqslant |\beta-\gamma| \leqslant \tfrac{2}{3}, \quad \tfrac{1}{3} \leqslant |\gamma-\alpha| \leqslant \tfrac{2}{3}.$$

These inequalities, together with (8), easily lead to a contradiction. Hence $f(x)$ cannot be of the form (6).

Suppose therefore that $f(x)$ is of the form (7). Then we have, by (5), $\frac{8}{9}A_2 = \frac{2}{3}A_1$; *i.e.* $A_2 = \frac{3}{4}A_1$, so that

$$f(x) = A_1\{(x_1+\alpha x_2+\beta x_3+\gamma x_4)^2+\tfrac{3}{4}(x_2^2+x_3^2+x_4^2+\tfrac{2}{3}x_2x_3-\tfrac{2}{3}x_2x_4+\tfrac{2}{3}x_3x_4)\}.$$

By taking $x_1 = x_3 = x_4 = 0$, $x_2 = 1$ we deduce that $\alpha^2+\frac{3}{4} \geqslant 1$ and hence, by (8), we have $\alpha = \frac{1}{2}$, and similarly $\beta = \gamma = \frac{1}{2}$. Thus we have

$$f(x) = A_1\{(x_1+\tfrac{1}{2}x_2+\tfrac{1}{2}x_3+\tfrac{1}{2}x_4)^2+\tfrac{3}{4}(x_2+\tfrac{1}{3}x_3-\tfrac{1}{3}x_4)^2+\tfrac{2}{3}(x_3+\tfrac{1}{2}x_4)^2+\tfrac{1}{2}x_4^2\} \quad (9)$$

$$= A_1(x_1^2+x_2^2+x_3^2+x_4^2+x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_3x_4). \quad (10)$$

It is easily checked that (9) is in fact of canonical form; we notice too that if we put $x_4 = 0$ we obtain a form equivalent to g_1.

Since f has unit determinant we must have $A_1 = \sqrt{2}$. By (10) it is clear that every unimodular transformation of $f(x)/\sqrt{2}$ yields a form with integral coefficients of $x_i x_j$ $(1 \leqslant i \leqslant j \leqslant n)$, and hence it follows that $M_2(f/\sqrt{2}) = \frac{3}{4}$, *i.e.* $M_2(f) = \frac{3}{2}$. This completes the proof of Theorem 5.

It is of interest to note that the form (9) possesses the property $A_4 = \frac{1}{2}A_1$. Korkine and Zolotareff (*op. cit.*) have shown that this property is only possessed by forms equivalent to (9).

From Theorems 4 and 5 and the known values

$$\gamma_1 = 1, \quad \gamma_2 = 2\,.\,3^{-1/2}, \quad \gamma_3 = 2^{1/3}, \quad \gamma_4 = 2^{1/2}, \quad \gamma_5 = 2^{3/5}, \quad \gamma_6 = 2\,.\,3^{-1/6},$$
$$\gamma_7 = 2^{6/7}, \quad \gamma_8 = 2,$$

(we write, as usual, γ_n for $\gamma_{n,1}$) it is possible to obtain upper bounds for $\gamma_{n,r}$ for small values of n and r. We state some of these results as

THEOREM 6. *We have*

(i) $\gamma_{3,2} = \gamma_{3,3} \leqslant 3\,.\,2^{-7/10}$, (ii) $\gamma_{6,2} = \gamma_{6,4} \leqslant \frac{9}{4}$,

(iii) $\gamma_{6,3} \leqslant 3^{9/10}\,.\,2^{-1/10}$, (iv) $\gamma_{7,2} = \gamma_{7,5} \leqslant 2^{12/7}\,.\,3^{-1/6}$,

(v) $\gamma_{7,3} = \gamma_{7,4} \leqslant 3^2\,.\,2^{-10/7}$, (vi) $\gamma_{8,2} = \gamma_{8,6} \leqslant 2^{12/7}$,

(vii) $\gamma_{8,3} = \gamma_{8,5} \leqslant 2^{17/7}\,.\,3^{-1/6}$, (viii) $\gamma_{8,4} \leqslant 3^2\,.\,2^{-6/7}$.

Proof. These come from Theorem 4 by taking the following values of n, r, m:

$$\text{(i) } n=5,\ r=4,\ m=2, \qquad \text{(ii) } n=6,\ r=4,\ m=2,$$

$$\text{(iii) } n=6,\ r=5,\ m=3, \qquad \text{(iv) } n=7,\ r=6,\ m=5,$$

$$\text{(v) } n=7,\ r=6,\ m=4, \qquad \text{(vi) } n=8,\ r=7,\ m=6,$$

$$\text{(vii) } n=8,\ r=7,\ m=5, \qquad \text{(viii) } n=8,\ r=7,\ m=4.$$

It is unlikely that any of the estimates are best possible.

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ON UNITARY EQUIVALENCE

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The problem has received some attention of determining the conditions that two square matrices, A, B should be equivalent for transformations under the unitary group, *i.e.*, that a matrix T can be found such that

$$\tilde{\bar{T}}AT = B, \quad \tilde{\bar{T}}T = I,$$

the symbols $-$ and $\sim$ denoting respectively complex conjugate and transposition.

Brenner† has shown how to form a canonical form for a matrix under the unitary group such that two matrices are unitarily equivalent if and only if they have the same canonical form. If a matrix A is transformed by a unitary matrix T, then $A\tilde{\bar{A}}$ is also transformed to $B\tilde{\bar{B}}$ by the same matrix T. But $A\tilde{\bar{A}}$ is Hermitian and may be transformed by further unitary transformation into a real diagonal matrix. This is Brenner's first step towards the canonical form. The matrix A may still be transformed by unitary matrices which commute with the diagonal matrix $A\tilde{\bar{A}}$. A sequence of rules to determine how these transformations may be used to effect a further simplification of A, leads to Brenner's definition of the canonical form by an inductive procedure.

Brenner's procedure, however, may be very complicated, and it is certainly not possible to visualize the general canonical form. In this paper a considerably simpler canonical form is described. Before proceed-

* Received 16 July, 1952; read 20 November, 1952.

† J. Brenner, " The problem of unitary equivalence ", *Acta Math.*, 86 (1951), 297–308.