

## 4400/6400 PROBLEM SET 7

Recommendation: 4400 students should do at least 4 problems; 6400 students should do at least 6.

7.1) Write out a careful proof of Lemma 6 in the Minkowski's Theorem handout. Especially, say a bit about the change-of-volume properties of the determinant map. Citing a reference or two might be appropriate.

7.2)\*\* Prove Theorem 11 in the Minkowski's theorem handout. (Possible strategy: talk to Professor Joseph H.G. Fu.)

7.3) Let  $\Omega \subset \mathbb{R}^N$  be a convex body with volume exactly  $2^N$ . (We have seen that  $\Omega$  need not have a nonzero lattice point.) Show that if  $\Omega$  is moreover *closed*, then it does have a nonzero lattice point.<sup>1</sup>

Suggestion: Use the fact that for all  $\epsilon > 0$ , the dilate  $(1 + \epsilon)\Omega$  must contain a nonzero lattice point.

7.4)\* Prove Theorem 12 in the Minkowski's Theorem handout.

7.5) Prove Lemma 14 (Euler's identity) in the Minkowski's Theorem handout.

7.6) Prove Lemma 16 in the Minkowski's Theorem handout.

7.7) The **three squares theorem** of Legendre-Gauss says that a positive integer  $n$  is a sum of three integral squares iff  $n$  is *not* of the form  $4^a \cdot (8k + 7)$ . This is significantly more difficult than the two and four squares theorems.

a) There exist positive integers  $a, b$  such that  $a$  and  $b$  are each sums of three squares but  $ab$  is not. Find the least such pair: say, with  $M = \max a, b$  smallest. (Don't worry:  $M < 20$ .) Deduce that the set of integers which are sums of three squares is not closed under multiplication, unlike the set of sums of two squares and sums of four (or more!) squares.

b) Show that the three squares theorem implies that every positive integer is a sum of three triangular numbers:  $\forall n \in \mathbb{Z}^+, \exists r, s, t \in \mathbb{Z} \mid n = \frac{r(r+1)}{2} + \frac{s(s+1)}{2} + \frac{t(t+1)}{2}$ .

7.8) One can deduce the four squares theorem from the three squares theorem. We outline an argument below. Your task is to fill in the steps.

Step 1: Let  $n \in \mathbb{Z}$ . If  $n \equiv 1, 2, 3, 5, 6 \pmod{8}$ , then  $n$  is a sum of three squares.

Step 2: If  $n \equiv 7 \pmod{8}$ , then  $n - 1 \equiv 6 \pmod{8}$ , so is a sum of three squares.

Step 3: Finally suppose that  $n \equiv 0, 4 \pmod{8}$ , so  $4 \mid n$ . Then by induction on  $n$ ,

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<sup>1</sup>Comment for the cognoscenti: thus we are assuming the **compactness** of  $\Omega$ . However the proof I suggest below does not use compactness in any way. I find this somewhat curious.

we may assume  $\frac{n}{4}$  is a sum of four squares, which implies  $n$  is a sum of four squares.

7.9) Which positive integers are the sum of four *positive* integral squares? (Some points will be awarded just for a correct statement of the result.)

In order to do the following problems, it will be helpful to read [Arithmetical Functions I: Multiplicative Functions].

7.10) Compute  $\omega(2009)$ ,  $\mu(2009)$ ,  $\varphi(2009)$ ,  $d(2009)$  and, for each  $k \geq 1$ ,  $\sigma_k(2009)$ . (Of course the answer to the last will be in terms of  $k$ .)

7.11) Find all  $n$  for which  $\varphi(n)$  is odd.

7.12) For each  $i$ ,  $1 \leq i \leq 10$ , find all positive integers  $n$  such that  $\varphi(n) = i$ .

7.13) What is the largest currently known perfect number?