

## 4400/6400 PROBLEM SET 6

Recommendation: 4400 students should do at least three problems; 6400 students should do at least four.

Heads up: There is an online applet for finding the fundamental solution to a Pell equation, available at

<http://www.numbertheory.org/php/pell.html>.

When asked for the fundamental solution  $u$  to a Pell equation you can use this applet, learn about continued fractions and try to find the solution yourself, or do some combination of the two.

6.0) Prove Lemma 1 of [Pell's Equation]: let  $(x, y)$  be a nontrivial integral solution to  $x^2 - Dy^2 = 1$ . Then:

- (i)  $x, y > 0 \iff x + \sqrt{D}y > 1$ .
- (ii)  $x > 0, y < 0 \iff 0 < x + \sqrt{D}y < 1$ .
- (iii)  $x < 0, y > 0 \iff -1 < x + \sqrt{D}y < 0$ .
- (iv)  $x, y < 0 \iff x + \sqrt{D}y < -1$ .

6.1) Find all integral solutions to the following equations:

- a)  $x^2 - 5y^2 = 1$ .
- b)  $x^2 - 53y^2 = 1$ .
- c)  $x^2 - 73y^2 = 1$ .
- d)  $x^2 - 1006009y^2 = 1$ .

- 6.2) a) Show that for any nonsquare positive integer  $d$  and any positive integer  $M$  there exist infinitely many integral solutions to the Pell equation  $x^2 - dy^2 = 1$  with  $y \equiv 0 \pmod{M}$ . (Suggestion: Write  $y = My'$  and "change variables.")
- b) Can we always find solutions with  $x \equiv 0 \pmod{M}$ ?

6.3) A **triangular number** is a positive integer of the form  $1+2+\dots+m = \frac{m(m+1)}{2}$ . A square number is (of course!) a number of the form  $n^2$ . A **square-triangular number** is a number which is simultaneously triangular and square, i.e., a solution of the equation

$$\frac{m(m+1)}{2} = n^2.$$

a) Show that the equation simplifies to

$$8n^2 = (2m+1)^2 - 1.$$

b) Substitute  $x = 2m+1$ ,  $y = 2n$  and show that solutions to  $x^2 - 2y^2 = 1$  correspond to square-triangular numbers, via

$$m = \frac{x-1}{2}, \quad n = \frac{y}{2}.$$

c) Use this correspondence to find all square-triangular numbers.

6.4) Let  $D \in \mathbb{Z}^+$ . The **negative Pell equation** is

$$x^2 - Dy^2 = -1.$$

When  $D$  is not a square, integral solutions correspond to units of norm  $-1$  in  $\mathbb{Z}[\sqrt{D}]$ . However, the issue of whether such units exist is more complicated.

- a) Find all solutions to the negative Pell equation when  $D$  is a square.
- b) Suppose that  $D$  is not a square and that the negative Pell equation has an integral solution. Show that  $D$  is not divisible by 4, nor by any prime of the form  $4k + 3$ .
- c)\* Find a nonsquare value of  $D$  satisfying the conditions of part b) for which the negative Pell equation nevertheless has no solutions.

Remark: It can be shown that the negative Pell equation (for nonsquare  $D$ ) has solutions iff the period length of the continued fraction expansion of  $\sqrt{D}$  is even. However this condition is awkward: it does not say much about the set of  $D$ 's for which there is a solution, and the computation for a given  $D$  can be extremely long.

6.5) For  $D$  a positive nonsquare and  $N$  any nonzero integer, one can consider the **generalized Pell equation**

$$x^2 - Dy^2 = N.$$

Suppose that this equation has a positive integer solution  $(x, y)$ . Show that it has infinitely many positive integral solutions  $(x_n, y_n)$ , where  $x_n + y_n\sqrt{D} = (x + y\sqrt{D})u^n$  and  $u$  is the fundamental solution to  $x^2 - Dy^2 = 1$ .

6.6G) Let  $D$  be a positive nonsquare integer. Show that the unit group  $\mathbb{Z}[\sqrt{D}]^\times$  of the ring  $\mathbb{Z}[\sqrt{D}]$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ . (Note that this is true whether or not there is any solution to the negative Pell equation, although what to take as the generator of the  $\mathbb{Z}$  factor will depend on this.)