

4400/6400 PROBLEM SET 6

Recommendation: 4400 students should do at least three problems; 6400 students should do at least four.

Heads up: There is an online applet for finding the fundamental solution to a Pell equation, available at

<http://www.numbertheory.org/php/pell.html>.

When asked for the fundamental solution u to a Pell equation you can use this applet, learn about continued fractions and try to find the solution yourself, or do some combination of the two.

6.0) Prove Lemma 1 of [Pell's Equation]: let (x, y) be a nontrivial integral solution to $x^2 - Dy^2 = 1$. Then:

- (i) $x, y > 0 \iff x + \sqrt{D}y > 1$.
- (ii) $x > 0, y < 0 \iff 0 < x + \sqrt{D}y < 1$.
- (iii) $x < 0, y > 0 \iff -1 < x + \sqrt{D}y < 0$.
- (iv) $x, y < 0 \iff x + \sqrt{D}y < -1$.

6.1) Find all integral solutions to the following equations:

- a) $x^2 - 5y^2 = 1$.
- b) $x^2 - 53y^2 = 1$.
- c) $x^2 - 73y^2 = 1$.
- d) $x^2 - 1006009y^2 = 1$.

6.2) a) Show that for any nonsquare positive integer d and any positive integer M there exist infinitely many integral solutions to the Pell equation $x^2 - dy^2 = 1$ with $y \equiv 0 \pmod{M}$. (Suggestion: Write $y = My'$ and "change variables.")

b) Can we always find solutions with $x \equiv 0 \pmod{M}$?

6.3) A **triangular number** is a positive integer of the form $1+2+\dots+m = \frac{m(m+1)}{2}$. A square number is (of course!) a number of the form n^2 . A **square-triangular number** is a number which is simultaneously triangular and square, i.e., a solution of the equation

$$\frac{m(m+1)}{2} = n^2.$$

a) Show that the equation simplifies to

$$8n^2 = (2m+1)^2 - 1.$$

b) Substitute $x = 2m+1$, $y = 2n$ and show that solutions to $x^2 - 2y^2 = 1$ correspond to square-triangular numbers, via

$$m = \frac{x-1}{2}, \quad n = \frac{y}{2}.$$

c) Use this correspondence to find all square-triangular numbers.

6.4) Let $D \in \mathbb{Z}^+$. The **negative Pell equation** is

$$x^2 - Dy^2 = -1.$$

When D is not a square, integral solutions correspond to units of norm -1 in $\mathbb{Z}[\sqrt{D}]$. However, the issue of whether such units exist is more complicated.

- a) Find all solutions to the negative Pell equation when D is a square.
- b) Suppose that D is not a square and that the negative Pell equation has an integral solution. Show that D is not divisible by 4, nor by any prime of the form $4k + 3$.
- c)* Find a nonsquare value of D satisfying the conditions of part b) for which the negative Pell equation nevertheless has no solutions.

Remark: It can be shown that the negative Pell equation (for nonsquare D) has solutions iff the period length of the continued fraction expansion of \sqrt{D} is even. However this condition is awkward: it does not say much about the set of D 's for which there is a solution, and the computation for a given D can be extremely long.

6.5) For D a positive nonsquare and N any nonzero integer, one can consider the **generalized Pell equation**

$$x^2 - Dy^2 = N.$$

Suppose that this equation has a positive integer solution (x, y) . Show that it has infinitely many positive integral solutions (x_n, y_n) , where $x_n + y_n\sqrt{D} = (x + y\sqrt{D})u^n$ and u is the fundamental solution to $x^2 - Dy^2 = 1$.

6.6G) Let D be a positive nonsquare integer. Show that the unit group $\mathbb{Z}[\sqrt{D}]^\times$ of the ring $\mathbb{Z}[\sqrt{D}]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. (Note that this is true whether or not there is any solution to the negative Pell equation, although what to take as the generator of the \mathbb{Z} factor will depend on this.)