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NOTE ON SAWYER'S PAPER "THE PRODUCT OF TWO NON-HOMOGENEOUS LINEAR FORMS"

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Let a non-homogeneous two-dimensional lattice Λ, that is, a lattice not containing the origin O, have determinant $\Delta' > 0$; and denote by $\mathfrak{R}$ the region

$$|xy| \leqslant \tfrac{1}{4}\Delta, \text{ where } \Delta > 0.$$

D. B. Sawyer gave recently† a very simple proof of Minkowski's theorem: that if no point of Λ lies in $\mathfrak{R}$, then $\Delta' > \Delta$. He considers a quadrilateral Q whose vertices are at four points of Λ, one in each quadrant, say $A(x_1, y_1)$, $B(-x_2, y_2)$, $C(-x_3, -y_3)$, $D(x_4, -y_4)$, where the x's and y's are positive. I remark that such a quadrilateral may be of two possible types: it may be convex, or non-convex with just one re-entrant angle, say at C. In both cases, two at least of the four triangles with vertices at three of A, B, C, D lie in Q, here certainly the triangles ACB and ACD. It follows that O is an inner point of Q; for O and A lie on the same side of CD, and O and C lie on the same side of AD, and if O is not in ACB then O and D lie on the same side of AC, and therefore O is in ACD.

Sawyer states without proof that there exists a quadrilateral Q which contains no point of Λ other than its four vertices. Now any triangle whose vertices are lattice points and which contains no other lattice point, has area $\frac{1}{2}\Delta'$, and therefore Q, as the sum of the areas ACB and ACD, has area Δ'. Since O is an inner point of Q, it follows that

$$\Delta' = \text{area } OAB + \text{area } OBC + \text{area } OCD + \text{area } ODA,$$

$$2\Delta' = (x_1 y_2 + x_2 y_1) + (x_2 y_3 + x_3 y_2) + (x_3 y_4 + x_4 y_3) + (x_4 y_1 + x_1 y_4)$$

$$> \tfrac{1}{2}\Delta + \tfrac{1}{2}\Delta + \tfrac{1}{2}\Delta + \tfrac{1}{2}\Delta,$$

* Received 5 November, 1952; read 20 November, 1952.

† *Journal London Math. Soc.*, 23 (1948), 250–251.

since $x_1 y_1 > \frac{1}{4}\Delta$ and $x_2 y_2 > \frac{1}{4}\Delta$, so that $x_1 y_2 + x_2 y_1 > \frac{1}{2}\Delta$, and so on. Thus $\Delta' > \Delta$.

The unproved assertion, however, requires consideration. It is not obvious, nor is it necessarily true, that replacing the vertices of Q by other lattice points inside Q, will lead to another quadrilateral containing no lattice point except at its vertices. But Minkowski's theorem will also follow if there exists a triangle such as ABD which contains O but contains no lattice point other than its vertices. For then

$$\tfrac{1}{2}\Delta' = \text{area } OAB + \text{area } OAD + \text{area } OBD,$$

$$\Delta' \geqslant (x_1 y_2 + x_2 y_1) + (x_1 y_4 + x_4 y_1) > \tfrac{1}{2}\Delta + \tfrac{1}{2}\Delta,$$

and again $\Delta' > \Delta$. We prove now that there exists either such a triangle or a quadrilateral Q of the desired kind.

Take B to be an arbitrary point of Λ in the second quadrant. The tangent from B to the first quadrant boundary $\mathcal{K}_1$ of $\mathcal{R}$ exists, and together with the line BO it bounds a space which contains in the fourth quadrant an arbitrarily large square. This will contain in its interior a point, say D, of Λ. Then BD does not intersect $\mathcal{K}_1$ and does not intersect the third quadrant. Take A to be an arbitrary point of Λ in the first quadrant. We show that we can replace the triangle ABD, say T, by a triangle $A'B'D'$, say T', contained in it, whose vertices are points of Λ in the appropriate quadrants and which contains no other point of Λ. Denote by $T(A)$ the part of the triangle T outside $\mathcal{R}$ in the first quadrant; then $T(A)$ is separated from BD by the tangent from B to $\mathcal{K}_1$. If A_1 is a point of Λ other than A in $T(A)$, we replace A by A_1 and obtain a triangle $A_1 BD$ which contains fewer lattice points than ABD. If A_2 is a lattice point in $T(A)$, we repeat the operation. Since there are only a finite number of lattice points in $T(A)$, and since each triangle is contained in the preceding one and contains fewer lattice points, we arrive at a lattice point A' such that $A'BD$ contains no lattice point other than A' in the first quadrant. We now repeat the process with B, observing that $T(B)$ is separated from $A'D$ by the y-axis. Similarly for D, noting that $T(D)$ is separated from $A'B'$ by the x-axis. We thus obtain the required triangle $A'B'D'$, and observe that A' still lies above and to the right of $B'D'$ and O still lies below and to the left of $B'D'$. It is possible that $B'D'$ may intersect $\mathcal{K}_1$, but the construction ensures that if so there will be no point of Λ in the area bounded by $B'D'$ and $\mathcal{K}_1$.

We now drop the dashes from A', B', D'. Take C to be any point of Λ in the third quadrant lying between the tangents from A to the second and fourth quadrant boundaries $\mathcal{K}_2$ and $\mathcal{K}_4$ of $\mathcal{R}$. Then $Q = ABCD$ is a convex quadrilateral, since C and A lie on opposite sides of BD and AC lies in the angle BAD. We can now apply the previous argument to B with reference to the triangle BAC, since any second quadrant lattice points in

BAC are separated from AC by the tangent from A to $\mathcal{K}_2$. Similarly for D. The quadrilateral Q is still convex, and contains no lattice points in the first, second or fourth quadrants other than A, B, D.

As remarked initially, O is an inner point of Q. If O lies in the triangle ABD, this triangle provides what is wanted. Now suppose that O lies in BCD. Then any third quadrant lattice points in Q lie in a region $T(C)$ which is separated from BD by (say) a parallel to BD through O. The usual argument now allows us to replace C by a lattice point C' in $T(C)$, giving a quadrilateral of the required kind. This completes the proof.

It should be noted that difficulty would have arisen in the last step if BD had intersected $\mathcal{K}_3$; for if there were lattice points in the region between them it might not have been possible to move C as desired. It should also be noted that the construction of the triangle ABD was such as to ensure, not only that this triangle itself contains no lattice point other than its vertices, but also that if BD intersects $\mathcal{K}_1$, then there is no lattice point in the region bounded by BD and $\mathcal{K}_1$. Were this not so, it might happen that however C were chosen there would be first quadrant lattice points in Q.

It may be noted that B. N. Delone*, in a Russian paper, has proved the stronger result: that if a non-homogeneous lattice has no point on the co-ordinate axes, then there exists a parallelogram whose vertices are lattice points, one in each quadrant, and which contains no other lattice point.

I am much obliged to Dr. Cassels and the referee for comments on my manuscript.

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* See *Mathematical Reviews*, 9 (1948), 334.