ON THE PRODUCT OF TWO NON-HOMOGENEOUS LINEAR FORMS

L. J. MORDELL*.

A well-known theorem of Minkowski states that, if L_1, L_2 are two homogeneous linear forms in the variables x_1, x_2 with real coefficients and determinant unity, and c_1, c_2 are any two real numbers, then a set of integer values of x_1, x_2, *i.e.* a lattice point (x), exists such that

$$|(L_1-c_1)(L_2-c_2)| \leqslant \tfrac{1}{4}. \tag{1}$$

Further the minimum value of the product is $\frac{1}{4}$ when and only when

$$(L_1-c_1)(L_2-c_2) \sim (x_1-\tfrac{1}{2})(x_2-\tfrac{1}{2}),$$

the sign of equivalence for the product referring to inhomogeneous linear substitutions with integer coefficients and of determinant ± 1.

Though a number of proofs of this result are known [Koksma, *Diophantische Approximationen* (Berlin, 1936), 19, contains references], the following may be of interest. The idea of this proof was also discovered independently by Davenport.

Let m^2 (where $m \geqslant 0$) denote the lower bound of the left-hand side of (1) for lattice points (x). If $m < \frac{1}{2}$ there is nothing to prove, and hence we may suppose that

$$m \geqslant \tfrac{1}{2}.$$

Then, for arbitrary $\epsilon > 0$, a lattice point (x') exists such that

$$|(L_1'-c_1)(L_2'-c_2)| = (m+\delta)^2, \tag{2}$$

for some δ with

$$0 \leqslant \delta < \epsilon. \tag{3}$$

On changing the origin, we may suppose that (x') is the origin O. Then

$$(m+\delta)^2 = |c_1 c_2| = c^2, \tag{4}$$

say, where $c > 0$. Then, from (4) and the definition of m, for all lattice points (x),

$$\left| \left(\frac{cL_1}{c_1}-c\right)\left(\frac{cL_2}{c_2}-c\right) \right| \geqslant m^2 = (c-\delta)^2.$$

* Received 26 October, 1940; read 17 December, 1940.

Since the forms cL_1/c_1, cL_2/c_2 have determinant unity, we may suppose by a slight change of notation that we are given two new homogeneous linear forms L_1, L_2, of determinant unity and with coefficients depending on ϵ, such that, for all lattice points (x),

$$|(L_1-c)(L_2-c)| \geqslant (c-\delta)^2, \tag{5}$$

where δ is some number satisfying (3) and depending only on ϵ and the coefficients of L_1, L_2.

We wish to show that, for sufficiently small ϵ, $c \leqslant \frac{1}{2}$, and to investigate when $c = \frac{1}{2}$. We consider the three possible cases when either $m > \frac{1}{2}$, or $m = \frac{1}{2}$, $\delta > 0$, or $m = \frac{1}{2}$, $\delta = 0$.

In the first two, I prove that there exists a lattice point P such that either P or $-P$ does not satisfy the inequality (5). The open parallelogram defined by

$$|L_1 - L_2| < 2c - 2\delta, \quad |L_1 + L_2| < 4c - 2\delta, \tag{6}$$

has area $$16(c-\delta)(c-\tfrac{1}{2}\delta) = 16m(m+\tfrac{1}{2}\delta) > 4,$$

and so, by Minkowski's classical theorem, it must contain a lattice point other than the origin, and this we shall take for P.

Since $L_1 \neq c$, $L_2 \neq c$, P must satisfy one of the following four sets of inequalities

$$L_1 > c,\ L_2 > c; \quad L_1 > c,\ L_2 < c; \quad L_1 < c,\ L_2 > c; \quad L_1 < c,\ L_2 < c. \tag{7}$$

The first three cases can be rejected. For we have respectively from (7) and (6),

$$L_1 - c + L_2 - c < 2c - 2\delta,\ \ L_1 - c + c - L_2 < 2c - 2\delta,\ \ L_2 - c + c - L_1 < 2c - 2\delta,$$

and so, from the inequality $2\sqrt{(AB)} \leqslant A+B$, (5) does not hold.

Similarly, for the first three of the four possible sets,

$$L_1 < -c,\ L_2 < -c; \quad L_1 < -c,\ L_2 > -c; \quad L_1 > -c,\ L_2 < -c;$$
$$L_1 > -c,\ L_2 > -c;$$

(6) shows that (5) cannot hold for the point $-P$.

Hence we are left with $|L_1| < c$, $|L_2| < c$. We can now suppose that P is not too near the origin O, really that, in (6), either $|L_1|$ or $|L_2|$ is greater than $\frac{1}{3}c$, say. For, if need be, (x) in (6) can be replaced by (kx), where k is a suitable integer. Then, since $c > \frac{1}{2}$, $L_1^2 + L_2^2 > 4c\epsilon$ and so

$$c^2 - L_1^2 + c^2 - L_2^2 < 2(c-\delta)^2,$$

whence $$(c^2-L_1^2)(c^2-L_2^2) < (c-\delta)^4,$$

i.e. (5) cannot hold for both P and $-P$.

We now consider the remaining case $m = \frac{1}{2}$, $\delta = 0$, $c = \frac{1}{2}$. We take instead of (6) the parallelogram

$$|L_1-L_2| \leqslant 1, \quad |L_1+L_2| \leqslant 2,$$

of area 4, containing a lattice point P other than O. In (7), we consider first the two cases

$$L_1 > \tfrac{1}{2},\ L_2 < \tfrac{1}{2}; \quad L_1 < \tfrac{1}{2},\ L_2 > \tfrac{1}{2};$$

and then (5) does not hold except possibly when

$$L_1-\tfrac{1}{2} = \tfrac{1}{2}-L_2 = \tfrac{1}{2},$$

i.e. $L_1 = 1$, $L_2 = 0$. But then the forms L_1, L_2 are equivalent to the forms $x_1+\lambda x_2$, x_2, where λ is a constant; and we may suppose $0 \leqslant \lambda < 1$. But if $\lambda \neq 0$

$$|x_1+\lambda x_2-\tfrac{1}{2}||x_2-\tfrac{1}{2}| < \tfrac{1}{4},$$

for $x_2 = 1$ and suitable x_1.

The only other case of (7) that need be considered is $L_1 > \frac{1}{2}$, $L_2 > \frac{1}{2}$, and then as before we need consider only $L_1-\frac{1}{2} = L_2-\frac{1}{2} = \frac{1}{2}$, *i.e.* $L_1 = 1$, $L_2 = 1$. But then the forms L_1, L_2 are equivalent to $x_1+\lambda x_2$, $x_1+(\lambda+1)x_2$, where $0 \leqslant \lambda < 1$. But if $\lambda \neq 0$

$$|x_1+\lambda x_2-\tfrac{1}{2}||x_1+(\lambda+1)x_2-\tfrac{1}{2}| < \tfrac{1}{4},$$

for $x_2 = 1$ and $|x_1+\lambda| < \frac{1}{2}$.

Since the case $L_1 < \frac{1}{2}$, $L_2 < \frac{1}{2}$ is disposed of above, this completes the proof.

The proof was suggested by geometric considerations and is essentially based on the fact that the four hyperbolas

$$|(x-c)(y-c)| \leqslant c^2, \quad |(x+c)(y+c)| \leqslant c^2,$$

enclose the parallelogram

$$|x-y| \leqslant 2c, \quad |x+y| \leqslant 4c,$$

whose sides $|x+y| = 4c$ each touch one hyperbola, while the sides $|x-y| = 2c$ each touch two of the hyperbolas.

The University,
Manchester.